Angular Momentum Transport in Stellar Interiors

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Abstract

Stars lose a significant amount of angular momentum between birth and death, implying that efficient processes transporting it from the core to the surface are active. Space asteroseismology delivered the interior rotation rates of more than a thousand low- and intermediate-mass stars, revealing the following:

- Single stars rotate nearly uniformly during the core-hydrogen and core-helium burning phases.
- Stellar cores spin up to a factor of 10 faster than the envelope during the red giant phase.
- The angular momentum of the helium-burning core of stars is in agreement with the angular momentum of white dwarfs.

Observations reveal a strong decrease of core angular momentum when stars have a convective core. Current theory of angular momentum transport fails to explain this. We propose improving the theory with a data-driven approach, whereby angular momentum prescriptions derived from
multidimensional (magneto)hydrodynamical simulations and theoretical considerations are continuously tested against modern observations. The TESS and PLATO space missions have the potential to derive the interior rotation of large samples of stars, including high-mass and metal-poor stars in binaries and clusters. This will provide the powerful observational constraints needed to improve theory and simulations.

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1. SETTING THE STAGE

Stars are radiating rotating gaseous spheres whose interior properties meet the laws of gravity and hydrodynamics. Understanding the properties of stellar interiors up to present-day observational precision requires a complex physical description involving various branches of physics and keeping in mind intense interactions between radiation and matter. Furthermore, the dynamics in stellar interiors and the life cycles of stars happen on length-scales and timescales that span so many orders of magnitude that we remain very far from full 3D stellar evolution computations, even with the use of the most powerful computers now or in the foreseeable future. We must therefore resort to 1D approximations when computing stellar evolution models.

Besides being fascinating in their own right, theoretical stellar structure and evolution models are basic ingredients for many areas of astrophysics. Indeed, stars are the building blocks of galaxies, clusters, associations, binaries, and exoplanetary systems. Therefore, these studies, in addition to stellar evolution theory, rely heavily on stellar models (and their accuracy). These models are constantly subjected to new observational constraints, as more, and particularly more precise, data become available. Such has happened with high-precision space photometry during the past decade, with pleasant confirmation of how well some aspects of stellar models represent reality, but with unanticipated surprises revealing how poor the models are on other aspects.

In this review, we focus on one pertinent aspect of stellar models that turns out to need major improvement: the angular momentum transport that occurs inside a star during its evolution. We treat this topic from three complementary aspects: modern observations, state-of-the-art theory, and multidimensional numerical simulations. Each of these aspects is detailed below with brief
historical reminders. The historical path for these three aspects is quite different in duration and maturity: Theory goes back almost a century, hydrodynamical simulations in 2D or 3D can only be done since the required computational power became available some 30 years ago, and direct observations of stellar interiors for stars other than the Sun from space astroseismology started about a decade ago.

When it comes to predicting the type of remnant at the end of stellar life—a white dwarf for low-mass stars or intermediate-mass stars born with a mass of $M \lesssim 8M_\odot$ and a neutron star or black hole for high-mass stars born with a mass of $M \gtrsim 8M_\odot$—the theory of stellar evolution is quite well established [e.g., Maeder (2009) and Kippenhahn et al. (2012) for modern monographs]. However, the theory of stellar interiors relies on physical concepts that are still subject to considerable uncertainties. Indeed, many of the (intrinsically multidimensional) phenomena connected with rotation, magnetism, mixing of chemical elements, and angular momentum transport inside stars cannot be deduced from first principles. Such physical ingredients therefore include one or more free parameters and these remained essentially uncalibrated prior to the astroseismology era, because observational measurements were mostly limited to constraints coming from the stellar photosphere.

Long-standing theoretical concepts taken for granted for stellar structure computations, such as local conservation of angular momentum, rotational mixing, and dynamical instabilities (see the extensive monographs by, e.g., Hansen et al. (2004) and Maeder (2009)) turn out to have unanticipated inaccuracies in terms of the transport of chemical elements and of angular momentum induced by them in the deep stellar interior. Theory and observations differ by two orders of magnitude with regard to core-to-envelope rotation rates in low-mass evolved stars, whereas the level of chemical mixing in young intermediate-mass stars connected with instabilities in their radiative layers adjacent to the core are orders of magnitude lower than anticipated. These limitations of the models are further outlined below. Many of these difficulties were only recently uncovered thanks to the probing capacity of stellar oscillations, offering a direct view of deep stellar interiors.

This review first offers a concise reminder of classical observational constraints from spectroscopy, interferometry, and astrometry of single, binary, and cluster stars used to calibrate stellar interiors after a brief discussion of the differences between 1D stellar evolution models with and without rotation. Such classical diagnostics at best have relative precisions on the order of a few percent but remain an important observational method to evaluate various types of stellar models for stars and stellar systems that do not reveal oscillations suitable for astroseismology. Next, we highlight the recent astroseismic input from stars that offer the opportunity to calibrate stellar models, with specific emphasis on internal rotation. Finally, we place the new observational diagnostics into context by providing a historical overview and the current status of the theory of transport processes in stars, detailing angular momentum transport in particular. We highlight the gain from a fully integrated approach combining observations, theory, and multidimensional simulations into a global picture. Such a comprehensive approach requires the application and integration of a multitude of techniques and methods based on observations and theory. An encompassing visualization is offered in Figure 1, where each of the keywords used to construct it is discussed below. It is a graphical representation of the conclusion of our review on the way forward toward a better understanding of angular momentum transport in stars.

1.1. One-Dimensional Stellar Evolution Models

As is outlined in Section 2, we now have astroseismic probes of the properties of stellar interiors. Such diagnostics give direct access to the physics at different depths inside the star, in particular the regions near the stellar core. Stellar models relying on physical descriptions that were taken
Figure 1

Synergies and complementarities between theory, various types of observations, and multidimensional simulations. Bridging all those aspects will lead to an inclusive methodology for 1D stellar modeling and a better understanding of stellar interiors, including angular momentum transport. Abbreviation: (M)HD, (magneto)hydrodynamical.

for granted for decades can now finally be calibrated asteroseismically. From this, it has been found that aspects of well-established theory do not meet the asteroseismic requirements, particularly in the case of rotation and angular momentum transport. In order to offer a path toward improved theory, we first remind the reader of some basic ingredients of stellar structure theory.

Stellar models are obtained by solving a set of differential equations based on the laws of physics, accompanied by proper boundary conditions for the center and surface of the star. Solving these equations requires that choices for the microscopic properties of stellar matter are made, in particular the equation of state of the gas, nuclear reactions, the interaction of radiation and matter, etc. Despite improved computational power over the past decades, stellar models necessarily remain a simplified version of reality, because it is not possible to compute complete 3D models across stellar evolution. Pragmatic choices in the approximations adopted to solve the stellar structure equations are thus in order. A major simplification is achieved when stellar rotation and stellar magnetism are ignored. Indeed, whenever the accompanying Coriolis, centrifugal, and Lorentz forces are neglected, spherical symmetry occurs. Furthermore, from a physical viewpoint, rotation and magnetism introduce a multitude of flows, waves, and instabilities, each of which with accompanying yet uncalibrated transport processes and exerting complex feedback on the stellar structure as time evolves (Maeder 2009, Mathis 2013). It is then obvious how to simplify the
stellar model computations in the first instance, namely by ignoring rotation and magnetism. These phenomena are therefore often only considered when 1D models fail to explain particular observational diagnostics.

How appropriate is it to ignore rotation and magnetism when computing stellar models? Both phenomena are closely related, and their importance for stellar evolution depends on the mass of the star. About 10% of intermediate- and high-mass stars have a detectable stable large-scale structured fossil magnetic field left from their formation and previous convective phases at the current threshold of CaII H & K absorption lines or spectropolarimetry based on Zeeman splitting (typically of a few Gauss; Donati & Landstreet 2009, Wade et al. 2016). This percentage agrees with the broad distribution of the measured spectroscopic projected equatorial surface rotation velocity, $v\sin i$ (e.g., Zorec & Royer 2012). The physical interpretation is that such stars are born without or with only a thin convective envelope, preventing the creation of a magnetic dynamo in their outer layers. In the absence of the latter, they do not experience a strong magnetized wind and therefore do not lose much angular momentum. Although stars born with a mass above some $\sim 15 \, M_\odot$ do experience a considerable radiation-driven wind (Kudritzki & Puls 2000) and lose angular momentum efficiently, most stars with $1.3 \lesssim M \lesssim 15 \, M_\odot$ roughly keep their original $J$ received at birth during the main sequence. Aside from these considerations for the stellar envelope, high- and intermediate-mass stars may have a magnetic dynamo in their convective core region. The strength and properties of such an interior field cannot be measured directly, as it is shielded by the extended overlying radiative envelope. Indirect evidence of the existence of interior fields, with strength above $\sim 10^5$ Gauss, was deduced from depressed dipole modes found in Kepler data of hundreds of red giants (Fuller et al. 2015, Cantiello et al. 2016). If these dipole modes undergo suppression by an interior magnetic field in the radiative core, then such a field must occur in about half of the progenitor main-sequence stars (Stello et al. 2016). However, Mosser et al. (2017) argue that mixed modes (i.e., modes with a g-mode character that are not pure p modes) are still detected in these stars, and, hence cannot be fully suppressed in their radiative core. Mosser et al. (2017) postulate the strongly reduced amplitudes of the dipole modes to be due to a yet unknown damping effect. Ongoing research approaches this topic from a complementary angle: The hypothesis that half of the progenitors of red giants must have a strong interior magnetic field in their convective core with properties derived by Cantiello et al. (2016), as deduced by Stello et al. (2016), is currently being tested from the effect of the acting Coriolis and Lorentz forces on the frequencies of detected gravity modes in intermediate-mass stars. Irrespective of the future outcome of this ongoing debate, the best approach for asteroseismology is to ignore core and surface magnetism in the computation of equilibrium models for the mass range of $1.3 \lesssim M \lesssim 15 \, M_\odot$ and to focus on rotation as the prime cause of complexity in 1D stellar models (e.g., Georgy et al. 2013) and in asteroseismology Buyschaert et al. (2018). Additional attention must be given to mass loss for intermediate-mass stars observationally (e.g., Barnes 2003).

### 1.1.1. One-dimensional stellar models without rotation and magnetism.

Even stellar evolution models representing nonrotating nonmagnetic stars have major challenges. A critical aspect is the type of energy transport within the various layers inside a star. In the case where energy transport by means of the diffusion of photons is sufficiently efficient, a radiatively stratified layer whose
MLT: mixing-length theory (of convection)

Mixing length ($\alpha_{\text{MLT}}$):
a free parameter representing the characteristic length scale over which convective fluid elements travel before they dissipate in their environment

Overshooting:
the phenomenon of turbulent convective fluid elements entering a radiative zone over an unknown distance, expressed as a fraction $\alpha_{\text{ov}}$ of the local pressure scale height

Thermal structure is determined by the radiative temperature gradient occurs. When photons are unable to transport the energy in particular stellar layers, these layers become dynamically unstable and macroscopic convective energy transport takes place. The general condition for convective stability is the Ledoux criterion:

$$\nabla_{\text{rad}} < \nabla_{\text{ad}} + \frac{\varphi}{\delta} \nabla_{\mu},$$

where we have introduced

$$\nabla_{\text{ad}} = \left( \frac{\partial \ln T}{\partial \ln P} \right)_S, \quad \nabla_{\mu} = \frac{\partial \ln \mu}{\partial \ln P}, \quad \delta = \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_{P,\mu}, \quad \text{and} \quad \varphi = \left( \frac{\partial \ln \rho}{\partial \ln \mu} \right)_{P,T},$$

with $\rho$ being the density, $P$ the pressure, $T$ the temperature, $S$ the entropy, and $\mu$ the mean molecular weight. In zones where $\nabla_{\mu} \neq 0$, the star is said to have built up a $\mu$-gradient. During core-hydrogen burning, this occurs due to the receding convective core for intermediate- and high-mass stars. For zones with a homogeneous chemical composition, the Ledoux criterion reduces to the Schwarzschild criterion:

$$\nabla_{\text{rad}} < \nabla_{\text{ad}}.$$  

The local angular frequency at which a fluid element oscillates when displaced within a convectively stable layer is the so-called Brunt–Väisälä frequency, or buoyancy frequency, $N(r)$. It can conveniently be written as a sum of two frequencies corresponding to the restoring of entropy ($N_T$) and of the chemical stratification ($N_r$), respectively:

$$N^2 = N_T^2 + N_r^2 = \frac{g}{H_P} \left[ \delta (\nabla_{\text{ad}} - \nabla) + \varphi \nabla_{\mu} \right],$$

with $g \frac{\partial \ln T}{\partial \ln P}$ being the local gravity and $H_P$ the local pressure scale height. This implies that the $\mu$-gradient zone left behind by a shrinking core affects the local behavior of $N(r)$. As is explained in Section 2, this affects the properties of stellar oscillations. Examples of $N(r)$ for different types of stars can be found in Supplemental Figures 1–4 of the Supplemental Text.

The timescale of the macroscopic convective motions in a star is negligible compared with the relevant timescales of stellar evolution. Therefore, turbulent convection in stellar models is often assumed to be time independent and treated by the mixing-length theory (MLT) of Böhm-Vitense (1958), though several alternatives exist (e.g., Canuto & Mazzitelli 1991). MLT assumes that the convective energy transport is taken care of by one single type of large-size eddy with one characteristic mean free path, called the mixing length. This parameter, here denoted as $\alpha_{\text{MLT}}$, is usually expressed in units of $H_P$ and takes typical values between 0.5 and 2.5, depending on the mass and metallicity of the star [see, e.g., Joyce & Chaboyer (2018) and Viani et al. (2018) for recent evaluations and calibration with respect to the solar value]. For low-mass main-sequence stars, $\alpha_{\text{MLT}}$ influences the size of the outer convective envelope. Intermediate-mass stars with $1.3 \lesssim M \lesssim 2M_\odot$ have thin convective envelope, and stars born with a higher mass mainly have convection in their fully mixed cores. The physical circumstances in these stars are very different from those in the Sun, hence $\alpha_{\text{MLT}}$ is essentially uncalibrated for such stars.

The full and instantaneous mixing caused by convection leaves a clear mark on the chemical history of the star. A major unknown in the theory of stellar interiors is the treatment of the physical conditions between the transition layers between convective and radiative zones. The thermal structure, the efficiency of chemical mixing, and the transport of angular momentum in such transition layers are unknown. Furthermore, the size and location of these transition layers are difficult to determine because they depend on the phenomenon of convective overshooting or penetration.
Owing to their turbulent motion and inertia, the fluid elements in a convective zone cannot stop abruptly when entering into a radiative layer. Their motion continues over an overshoot distance $\alpha_{ov}$ (expressed in $H_p$), into the transition layer. The treatment of the mixing in a core overshoot zone beyond the Schwarzschild boundary at radial coordinate $r_{cc}$, here denoted as $D_{mix}(r > r_{cc})$, is of critical importance because it determines how much fuel takes part in the nuclear reactions. In particular, the amount of helium/carbon at the end of the core-hydrogen/helium burning, depends on the structure, hydrodynamics, and mixing efficiency in the overshoot zone. The observational estimation of the properties of the core overshoot zone is of major importance, because these determine the core mass during the evolution of the star. Convective envelope overshooting (a.k.a. undershooting) may occur as well, but its effect on the star’s evolution is small compared to the one of core overshooting.

Zahn (1991) provided a coherent physical picture of the processes of overshooting and/or penetration. First, turbulent stellar convection zones are the seed of large-scale coherent structures commonly referred to as plumes. Because of their inertia, they penetrate into adjacent stellar radiative zones. The key physical control parameter is the Péclet number defined as $Pe = v_c l_c / K_T$, where $v_c$ and $l_c$ are the characteristic convective velocity and length scale, respectively, and $K_T$ is the thermal diffusivity. Two regions are identified. In the first one, $Pe > 1$, i.e., the flow dynamics, is driven by the advection and the plume keeps its identity. The plumes penetrate the stable radiative region and they render it nearly adiabatic over a penetration distance $d_{pen}$, whereas they are decelerated by the buoyancy force. When the Péclet number drops below unity, $Pe < 1$, thermal diffusion operates faster than advection, and the temperature gradient $\nabla$ adjusts from adiabatic to radiative in an overshoot region with thickness $d_{oven}$.

To guide stellar modelers in their quest to describe convective penetration/overshoot, Viallet et al. (2015) proposed three regimes. First, for plumes with $Pe \leq 1$, which are only able to mix composition without affecting the entropy structure, an exponential diffusion coefficient as proposed by Freytag et al. (1996) can be adopted. For plumes with $Pe \geq 1$, the entropy and the chemicals are both efficiently mixed, and one can assume that $\nabla = \nabla_{ad}$; i.e., the chemical composition is assumed to be homogeneous owing to instantaneous and full mixing. Finally, for plumes with $Pe \gg 1$, turbulent entrainment of mass by convective flows occurs (Fernando 1991). Another way to quantify convective penetration and overshooting is by numerical simulations, where strong efforts have been undertaken to follow the whole coverage in Péclet number (e.g., Browning et al. 2004, Meakin & Arnett 2007, Rogers et al. 2013, Viallet et al. 2013, Brun et al. 2017).

Nowadays, the overshoot region can be probed observationally by investigating its effect on observed and identified oscillation frequencies (e.g., Deheuvels et al. 2016, Constantino et al. 2017, Pedersen et al. 2018). This is a promising way to constrain the physical properties at the boundaries of convective and radiative layers. Asteroseismology allows the derivation of $D_{mix}(r > r_{cc})$, including the overshooting zones, and determination of the dependence of convective penetration/overshoot on rotation and magnetic fields.

### 1.1.2. One-dimensional stellar models with rotation.

A large fraction of early-F to O stars are fast rotators (Zorec & Royer 2012). Observations of such stars, notably surface abundances and color-magnitude diagrams of clusters, triggered the development of stellar evolution models for rotating stars (e.g., Maeder & Meynet 2000, Brott et al. 2011, Ekström et al. 2012). These usually adopt the approximation of shellular rotation (Zahn 1992), for which the shape of the star is treated in the framework of the Roche model, with the polar and equatorial radii differing maximally by a factor of 1.5 at the critical rotation rate $\Omega_{crit} \equiv \sqrt{8GM/27R_p^3}$, where $R_p$ is the polar radius of the star (see the sidebar titled Relevant Frequencies in Stellar Interiors).
RELEVANT FREQUENCIES IN STELLAR INTERIORS

- $\Omega_{\text{surf}}$: angular surface rotation frequency of a star
- $\Omega_{\text{env}}$: average angular rotation frequency in the stellar envelope
- $\Omega_{\text{core}}$: average angular rotation frequency in the core region of the star
- $\Omega_{\text{crit}}$: Roche critical angular rotation frequency of the star defined as $\sqrt{8GM/27R_p^3}$ with $R_p$ being the polar radius
- $\omega_{\text{nlm}}$: angular frequency of an oscillation mode with degree $l$, azimuthal order $m$, and radial order $n$
- $N(r)$: local buoyancy (or Brunt–Väisälä) frequency
- $S_l(r)$: characteristic acoustic (or Lamb) frequency of a mode with degree $l$
- $\omega_{\Lambda}(r)$: the Alfvén frequency, i.e., the orbital frequency of a binary

Rotation produces latitudinal dependence of the radiative flux, of the effective temperature, of the stellar wind, etc. A basic assumption relied upon to compute models of rotating stars is that the angular momentum lost from the outer envelope at each time step, due to a stellar wind, implies a change in the angular velocity distribution inside the star. Furthermore, as the core contracts and the envelope expands during the evolution, the rotation profile changes. This introduces a myriad of flows and instabilities, causing chemicals to mix and angular momentum to get transported in the radiative zones of stars. Extensive theory has been developed to describe rotationally induced instabilities and their accompanying mixing processes, both in low-mass stars (e.g., Chaboyer et al. 1995) and in intermediate- and high-mass stars (e.g., Talon et al. 1997, Pinsonneault 1997, Heger et al. 2000). These processes are sometimes implemented with free parameters in evolution codes, e.g., representing the level of chemical mixing and/or angular momentum transport. The transport processes are subject to considerable uncertainties because they cannot be properly calibrated by classical observations, and few have been tested against simulations. With the advent of asteroseismology, the theory can finally be evaluated. As an example, we confront below the theoretical prediction that $\Omega_{\text{core}}/\Omega_{\text{surf}} \in [1, 8]$ for main-sequence stars with $1.7 \lesssim M \lesssim 15 M_\odot$ (e.g., Georgy et al. 2013, their figure 7).

1.2. Classical Observational Constraints for Stellar Interiors

High-resolution high signal-to-noise spectroscopy is a major classical observational method for evaluating stellar evolution models. Spectroscopy allows estimation of the effective temperature ($T_{\text{eff}}$), gravity ($\log g$), projected surface rotation $v \sin i \equiv \Omega_{\text{surf}} R \sin i$ with $R$ being the stellar radius, and photospheric abundances of stars. The relative precisions of these spectroscopic diagnostics, which are widely available for stars in the Milky Way and Magellanic Clouds, are compared with other diagnostics for stellar interiors in Table 1. Estimation of values ($T_{\text{eff}}$, $\log g$, $v \sin i$) depends on spectrum normalization, particularly of hydrogen lines, and suffers from degeneracies. Despite this, relative uncertainties for $T_{\text{eff}}$ may reach the level of 1% for low-mass stars. For high-mass stars, one usually cannot do better than 5% owing to the limited number of spectral lines. Relative uncertainties for $\log g$ are worse owing to degeneracies and are therefore not included in Table 1. Estimation of $v \sin i$ only offers indirect information due to the unknown values $\sin i$ and $R$. Furthermore, $v \sin i$ is also subject to uncertainties caused by other spectral line broadening phenomena (e.g., Aerts et al. 2014b, Simón-Díaz & Herrero 2014). Photospheric abundances deliver a powerful spectroscopic constraint and have been used extensively to evaluate stellar evolution theory. Low-mass stars in the Solar Neighborhood can be analyzed differentially with respect to the Sun, such that systematic uncertainties cancel out. For these, the relative errors of $[\text{Fe/H}]$
Table 1  Observational diagnostics used to calibrate models of stellar interiors and their optimal relative precision

<table>
<thead>
<tr>
<th>Method</th>
<th>Type of star</th>
<th>Diagnostic</th>
<th>Precision</th>
<th>Model dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectral lines</td>
<td>LM</td>
<td>$T_{\text{eff}}$, abundances</td>
<td>$\sim 1%$</td>
<td>A: medium</td>
</tr>
<tr>
<td>Spectral lines</td>
<td>IM</td>
<td>$T_{\text{eff}}$, abundances</td>
<td>$\sim 2%$</td>
<td>A: medium</td>
</tr>
<tr>
<td>Spectral lines</td>
<td>HM</td>
<td>$T_{\text{eff}}$, abundances</td>
<td>$\sim 5%$</td>
<td>A: medium</td>
</tr>
<tr>
<td>IRFM/SEDh</td>
<td>LM, IM</td>
<td>$T_{\text{eff}}$</td>
<td>$\sim 2%$</td>
<td>A: low</td>
</tr>
<tr>
<td>RV &amp; light curvesc</td>
<td>EB/SB2</td>
<td>$M$</td>
<td>$\sim 1%$</td>
<td>None</td>
</tr>
<tr>
<td>RV &amp; light curvesc</td>
<td>EB/SB2</td>
<td>$R$</td>
<td>$\sim 3%$</td>
<td>A: low</td>
</tr>
<tr>
<td>Interferometryd</td>
<td>All</td>
<td>$R$</td>
<td>$\sim 3%$</td>
<td>A: low</td>
</tr>
<tr>
<td>Typical Gaia DR2e</td>
<td>LM &amp; IM</td>
<td>$L$</td>
<td>$\lesssim 15%$</td>
<td>A: medium</td>
</tr>
<tr>
<td>Typical Gaia DR2e</td>
<td>LM &amp; IM</td>
<td>$R$</td>
<td>$\lesssim 10%$</td>
<td>A: medium</td>
</tr>
<tr>
<td>Cluster (E)MSTf</td>
<td>All</td>
<td>age</td>
<td>$\sim 30%$</td>
<td>I: strong</td>
</tr>
<tr>
<td>Gyrochronologyg</td>
<td>LM</td>
<td>$\Omega_{\text{surf}}$</td>
<td>$\sim 10%$</td>
<td>None</td>
</tr>
<tr>
<td>Gyrochronologyg</td>
<td>LM</td>
<td>age</td>
<td>$\sim 20%$</td>
<td>I: medium</td>
</tr>
<tr>
<td>Coherent g modesh</td>
<td>IM</td>
<td>$\omega_{\text{dim}}$</td>
<td>$\sim 0.1%$</td>
<td>None</td>
</tr>
<tr>
<td>Coherent p modesh</td>
<td>IM</td>
<td>$\omega_{\text{dim}}$</td>
<td>$\sim 0.01%$</td>
<td>None</td>
</tr>
<tr>
<td>Damped p modesh</td>
<td>LM</td>
<td>$\omega_{\text{dim}}$</td>
<td>$\sim 0.001%$</td>
<td>None</td>
</tr>
<tr>
<td>Damped mixed modesh</td>
<td>RG</td>
<td>$\omega_{\text{dim}}$</td>
<td>$\sim 0.01%$</td>
<td>None</td>
</tr>
<tr>
<td>g-mode splittingsh</td>
<td>IM</td>
<td>$\Omega_{\text{core}}$</td>
<td>$\sim 0.1%$</td>
<td>None</td>
</tr>
<tr>
<td>g-mode spacingsh</td>
<td>IM</td>
<td>$\Omega_{\text{core}}$</td>
<td>$\sim 5%$</td>
<td>I: low</td>
</tr>
<tr>
<td>p-mode splittingsh</td>
<td>IM</td>
<td>$\Omega_{\text{env}}$</td>
<td>$\sim 30%$</td>
<td>I: medium</td>
</tr>
<tr>
<td>p-mode splittingsh</td>
<td>LM</td>
<td>$\Omega_{\text{env}}$</td>
<td>$\sim 50%$</td>
<td>I: medium</td>
</tr>
<tr>
<td>mixed-mode splittingsh</td>
<td>RG</td>
<td>$\Omega_{\text{core}}$</td>
<td>$\sim 1%$</td>
<td>None</td>
</tr>
<tr>
<td>Phase modulation &amp; RVh,i</td>
<td>PB1/PB2</td>
<td>$M, R$</td>
<td>as EB/SB2</td>
<td>None</td>
</tr>
</tbody>
</table>

Abbreviations: All, all masses; EB/SB2, double-lined eclipsing binaries; DR, data release; HM, high mass; IM, intermediate mass; IRFM/SED, infrared flux method or spectral energy distribution; LM, low mass; PB1/PB2, binaries with one or two pulsating components; RG, red giant; RV, radial velocity.

1We indicate in the column the diagnostic’s dependence on stellar atmosphere (A) and/or stellar interior models (I).

1Provided that a good estimate of the reddening and an absolute flux calibration are available; this method can also lead to a radius estimate at a level of $\sim 3\%$ when a high-precision parallax (at the level of Gaia DR3) becomes available.

1Modeling of double-lined eclipsing binaries (EB/SB2) based on RVs from spectroscopy and light curves from photometry, with both covering the orbit of the binary (Torres et al. 2010).

1Requires good calibration stars, as well as a high-precision parallax and $T_{\text{eff}}$, which limits applicability to bright stars.

1Typical values covering 77 million stars (Andrae et al. 2018).

1Isochrone fitting of (extended) main-sequence turnoffs, (E)MST, of clusters from combined multicolor photometry, spectroscopy, and astrometry (Bastian & Lardo 2017).

1From micromagnitude-precision time-series space photometry covering many cycles of the rotationally modulated light curve due to spots, delivering $\Omega_{\text{surf}}$.

1Requires long-duration micromagnitude-precision uninterrupted space photometric light curves, delivering the frequencies $\omega_{\text{dim}}$ of tens of identified nonradial oscillation modes $(n, l, m)$, where we additionally note that combined $T_{\text{eff}}$, Gaia DR2, and damped modes lead to radii with $\sim 1\%$ precision for the best cases (Zinn et al. 2018).

1The potential of this method is largest for p modes and depends on the nature of the pulsating binary (one or two pulsating components), where PB2/SB2 have similar potential as EB/SB2 [cf. Murphy et al. (2016) for details and first applications].

may be as low as $1\%$ to $2\%$ (e.g., Meléndez et al. 2014, Nissen 2015). In the case of high-mass stars, the relative errors for [C/H], [N/H], [O/H], and [Fe/H] are typically at least $5\%$ to $10\%$ (Morel et al. 2008, Przybilla et al. 2013, Martins et al. 2015). A positive correlation between the surface nitrogen abundance and $v \sin i$ has been established for about half of the OB-type stars [see, e.g., Dufton et al. (2018) for a recent discussion]. However, the nitrogen abundance turned out to be uncorrelated with the measured surface rotation frequency, $\Omega_{\text{surf}}$, and magnetic field for a sample of 68 slowly to moderately rotating B-type field stars (Aerts et al. 2014a). Rather, evidence for a weak correlation between the dominant oscillation mode frequency and the measured nitrogen abundance was found, suggesting pulsational mixing in this sample. Recent 2D hydrodynamical
Simulations of convectively driven internal gravity waves (IGWs) for a 3-\(M_\odot\) star indeed predict efficient particle mixing in radiative layers (Rogers & McElwaine 2017).

Double-lined spectroscopic eclipsing binaries have long been known as excellent calibrators for stellar models. Indeed, spectroscopic and (multicolor) photometric data covering their orbit offers model-independent stellar masses at the level of 1–3\% from the binary motion [see Torres et al. (2010) for an extensive review based on 95 objects], covering the mass range from 0.2 to 27 \(M_\odot\). Kepler space photometry revealed similar potential for noneclipsing double-lined binaries pulsating with coherent p modes from the phase modulation method (Murphy et al. 2016). Double-lined eclipsing binaries have also been used to estimate convective core overshooting, suggesting a mass dependence (Claret & Torres 2018). However, core overshooting depends strongly on various other stellar parameters, and such correlations should be considered in the binary modeling. The first steps to do so were taken by Johnston et al. (2019), who introduced the concept of isochrone-cloud fitting as an alternative to classical isochrone fitting for binaries (clusters) in which one limits to the case that all stars have the same overshoot and mixing properties.

Interferometric data offers good estimates of angular diameters, which can be translated into precise stellar radii to evaluate stellar models for the brightest stars. This technique offers a level of precision for \(R\) of \(\sim 3\%\) for nearby stars, provided that a good parallax is available (e.g., Ligi et al. 2016). Two major restrictions with this method are the limited availability of good calibrators and uncertainties in the limb darkening (e.g., White et al. 2018). In addition to interferometry of bright nearby stars, \(Gaia\) astrometry offers the potential to derive accurate radii for millions of stars in the Milky Way, whenever a good estimate of \(T_{\text{eff}}\) is available. \(Gaia\) DR2 (data release 2) already led to radii at \(\sim 2\%\) precision for many (exoplanet host) stars (e.g., Fulton & Petigura 2018) and even \(\sim 1\%\) for asteroseismic targets (Zinn et al. 2018). At the level of \(Gaia\) DR3 (data release 3, scheduled for 2021), stellar radii with a precision of \(\sim 1\%\) are anticipated as the norm for millions of single and binary stars.

Isochrone fitting of color-magnitude diagrams of stellar clusters has been used extensively to test models of stellar interiors, by deducing stellar ages from the observed turnoff point. Large observing campaigns with the \(Hubble\) Space Telescope led to systematic detections of multiple main sequences in globular clusters covering masses from 0.1 to 1.8 \(M_\odot\) and ages from 2 to 15 Gyr (e.g., Piotto et al. 2002, Dotter et al. 2007). These are interpreted as due to different populations of stars with abundance variations indicative of nuclear processing by the CNO cycle [see, e.g., Piotto et al. (2007) and Bastian & Lardo (2017) for early discoveries and an extensive review, respectively]. By contrast, open clusters with intermediate- and high-mass stars in the Milky Way and Magellanic Clouds reveal extended main-sequence turnoffs and/or split main sequences. Interpretation of these in terms of stellar models with rotation leads to ages in agreement with the lithium depletion method (e.g., Cummings & Kalirai 2018). It remains unclear what level of the extended turnoffs is due to rotation (or other phenomena) and what level is due to a spread in age (e.g., Bastian et al. 2018, Gossage et al. 2018). Interpretation of the observed extended turnoffs as solely due to age leads to about 30\% uncertainty for stellar aging. \(Gaia\) DR2 luminosities (\(L\)) led to spectacular improvements of HR diagrams of clusters (Gaia Collab. et al. 2018) and \(Gaia\) DR3 is anticipated to bring more potential for the improvement of stellar interiors from fitting of the overall cluster morphologies, keeping in mind the option of different mixing properties for each of the cluster stars (Johnston et al. 2019).

Another way to derive stellar age as an observational test of stellar models is by the method of gyrochronology. Prior to high-precision space photometry, it was realized that the presence of an envelope dynamo in low-mass stars offers a good way of aging them (Barnes 2003). Gyrochronology is based on the stellar spin-down due to magnetic breaking and uses the surface rotation rate as a clock, adopting the so-called Skumanich relation between angular momentum loss and rotation...
rate, $dJ/dr \propto \Omega_{\text{surf}}$ (Skumanich 1972). This has meanwhile been extensively applied to Kepler data of low-mass (exoplanet-hosting) field and cluster stars (García et al. 2014, Meibom et al. 2015). Latitudinal surface differential rotation limits gyrochronology for field stars to the level of $\sim$2 Gyr, i.e., some 20% relative precision (Epstein & Pinsonneault 2014). Furthermore, the occurrence of anomalously rapid rotation in old field stars hints at weakened magnetic breaking at evolved stages of low-mass stars (van Saders et al. 2016). This implies that more complex models than a simple empirical relation between the rotation rate and stellar age are required for precise stellar aging from gyrochronology of low-mass stars. Such models can be designed by calibrating gyrochronology with asteroseismically determined ages (García et al. 2014, Ceillier et al. 2016). This brings us to the recent breakthroughs in stellar modeling based on asteroseismology, keeping in mind the broad availability of classical diagnostics to evaluate stellar interiors summarized in Table 1.

2. THE NEW AGE OF ASTEROSEISMOLOGY

2.1. Asteroseismology in a Nutshell

Stellar oscillations (a.k.a. starquakes) offer a direct probe of stellar interiors. Asteroseismology, the interpretation of detected identified oscillation modes (Aerts et al. 2010), offers a unique way to evaluate and calibrate stellar interiors. It uses the measured high-precision frequencies of stellar oscillations as revealed by time-domain astronomy as an observational diagnostic of the stellar interior to evaluate stellar models, in addition to classical observables (Table 1). Asteroseismology is the “champion” in the observational probing of stellar interiors. (See the sidebar titled The Plethora of Coherent and Damped Linear Oscillation Modes Inside Stars.)

THE PLETHORA OF COHERENT AND DAMPED LINEAR OSCILLATION MODES INSIDE STARS

Whenever a spherically symmetric star in hydrostatic equilibrium gets perturbed, it may become unstable. From a theoretical viewpoint, small disturbances of the equilibrium configuration are treated in a linear approach. As such, the reactions of the acting forces in the equation of motion are investigated to understand and interpret stellar (in)stability (Ledoux 1941, Unno et al. 1989). The instabilities may give rise to detectable self-driven coherent oscillation modes with long lifetimes (thousands to millions of years) whenever their growth rate is positive and when the thermal timescale of the excitation layer in the stellar envelope is longer than the period of the oscillation modes (e.g., Cox 1980, Pamyatnykh 1999). But even when modes are damped, they may still be excited to observable amplitudes by stochastic forcing. This is the case for stars with an outer convective envelope, such as the Sun (Christensen-Dalsgaard 2002) and red giants (Chaplin & Miglio 2013), and gives rise to oscillation modes with lifetimes of days to weeks.

Perturbation of the linearized stellar structure equations leads to the properties of the oscillations of a particular star, among which are its mode frequencies. The modes are categorized according to the dominant restoring force. Pressure (or acoustic), gravity, inertial, and magnetic (or Alfvén) oscillations occur when the dominant restoring force is the pressure force, gravity (buoyancy), the Coriolis force, and the Lorentz force, respectively. When stars are hosting close planetary or stellar companions, tidal forces may trigger these different types of modes; in this case they are called tidal oscillations. In practice, the pressure force and gravity are always active, whereas the other forces may be of minor importance. Hence, one has introduced the specific terminology of $p$ (for pressure) and $g$ (for gravity) modes. In reality, the forces act together to restore the equilibrium of the star, and one may encounter the situation in which two (or more) among them are of similar importance. In that case, the distinction of the modes in terms of frequency may become less clear, and the terminology is adapted accordingly, e.g., gravito-inertial modes, magneto-acoustic modes, etc.
Asteroseismology follows the foundations laid out for the interpretation of the solar oscillations by means of helioseismology (Christensen-Dalsgaard 2002). The application of this method to distant stars was made possible thanks to uninterrupted high-precision photometric time-series data assembled by recent space missions. In this review, we mainly rely on light curves assembled by the NASA \textit{Kepler} telescope (Koch et al. 2010) and its refurbished K2 version (Howell et al. 2014). These data have a precision of \(\sim 1\) \(\mu\)mag, cover up to four years and three months, respectively, and offer the long-awaited evaluation of the theory of stellar interiors. It turns out that shortcomings in the theory of angular momentum transport already occur on the main sequence, impacting all subsequent phases (e.g., Eggenberger et al. 2017, Tayar \& Pinsonneault 2018, Townsend et al. 2018). As such, time-domain space photometry assembled during the past decade is a game changer in the calibration of stellar structure and evolution theories.

Asteroseismic probing of stellar interiors is achieved by interpreting the properties of detected nonradial oscillation modes, notably their frequency. Gravity modes allow one to probe the deepest layers of stars, where chemical gradients are built up during stellar evolution. In contrast, the p modes probe the outer envelope of stars. The oscillation modes can often be represented by simple waves, whose angular frequencies \(\omega\) fulfill dispersion relations [see Smeyers \& Van Hoolst (2010) for a summary of mathematical treatments]. As an example, p modes correspond to acoustic waves and g modes to IGW. For acoustic waves, the dispersion relation involves the sound speed of the gas, whereas for gravity waves it is the buoyancy frequency, \(N\). Although pressure waves can propagate in both convective and radiative regions, gravity waves can only propagate in convectively stable radiative layers.

The wave propagation regions are fully determined by the interplay between \(N(r)\) and the so-called local characteristic acoustic frequency for the mode with degree \(l\):

\[
S_l^2(r) = \frac{l(l + 1)c^2}{r^2},
\]

where \(c\) is the sound speed. \textbf{Supplemental Figures 1–4} show wave propagation diagrams for various types of intermediate-mass stars in the case of dipole modes. Such diagrams illustrate the cavities in which the waves can propagate inside the star and, thus, where the corresponding modes have probing power. The modes are said to be evanescent in regions where the solutions to the perturbed stellar structure equations are exponentially decreasing instead of oscillatory [cf. Aerts et al. (2010) for a thorough description]. Just as oscillation modes can be coherent or damped in nature, the corresponding waves can have a standing or traveling wave nature in their propagation cavities. A broad spectrum of both damped and standing IGWs can be generated by partial ionization layers in radiative envelopes and/or by stochastic forcing at convective-radiative interfaces. Although stochastic damped IGWs cannot be used for stellar modeling like their coherent counterparts, unless they coincide with eigenmodes of the star, they do transport angular momentum.

For an extensive discussion on the theory of nonradial oscillation modes, their excitation, their relation to waves, and their propagation, we refer to the monographs by Unno et al. (1989), Smeyers \& Van Hoolst (2010), and Aerts et al. (2010). The latter book also offers methodology for the frequency analysis and identification of nonradial oscillation modes in detail. Here, we limit the discussion to the bare minimum necessary to understand the exploitation of detected oscillation modes and IGWs in terms of the stellar interior, with emphasis on rotation. For modern descriptions of coherent mode excitation, we refer to Dupret et al. (2005), Bouabid et al. (2013), and Szewczuk \& Daszyńska-Daszkiewicz (2017). IGW generation by convective cores, convective
envelopes, or thin convection zones due to opacity bumps in radiative envelopes is extensively discussed by Rogers et al. (2013), Talon & Charbonnel (2008), and Cantiello et al. (2009), respectively.

Unlike ground-based time-series data with low duty cycle, high-cadence uninterrupted space photometry does not lead to daily aliasing confusion in the derivation of the oscillation frequencies. As such, the frequency uncertainty is dominated by the frequency resolution of the data set. For coherent modes, this resolution is $\sim 1/T$, with $T$ the total time base of the time-series data. This is $0.00068 \, \text{d}^{-1}$ ($0.0079 \, \mu\text{Hz}$) for the four-year nominal Kepler data. For damped modes, the resolution is $\sim 1/\sqrt{T}$. Aside from the resolution of the data, the frequency error of each detected mode depends on its amplitude and the noise properties of the data in the relevant frequency regime (Aerts et al. 2010, chapter 5). Damped modes have similar amplitudes and allow their frequencies to be deduced from peak bagging (e.g., Lund et al. 2017). By contrast, coherent modes may have mode amplitudes differing by six orders of magnitude, and their frequencies must therefore be deduced from a prewhitening procedure (e.g., Aerts et al. 2010, chapter 5). Iterative prewhitening inherently introduces uncertainties on the order of the frequency resolution of the data. The nominal four-year Kepler data lead to a relative frequency precision better than 0.1% for coherent g modes and 0.01% for coherent p modes in intermediate- and high-mass stars. Damped p modes of low-mass stars can be measured with a typical relative precision better than 0.001% for dwarfs (Lund et al. 2017) and 0.01% for red giants (Hekker & Christensen-Dalsgaard 2017, Mosser et al. 2018).

Exploitation of such ultraprecise asteroseismic information has brought a wealth of information on stellar interior processes that is inaccessible from classical data. We dedicate this review to the asteroseismic inference of interior rotation as it constrains internal angular momentum transport. As is explained below, the asteroseismic estimation of the rotation rate as a function of depth $r$ inside the star, $\Omega(r)$, can be fully or quasi–model independent. Which of the two it is depends on the kind of star, the nature of the detected oscillations, and the level of stellar rotation, as explained in Section 2.4. First, we discuss how an asteroseismically calibrated model of a star can be sought from its identified modes before moving on to estimation of the interior rotation. (See the sidebar titled Spherical Harmonics and Rotational Splitting in Asteroseismology.)

**Spherical Harmonics and Rotational Splitting in Asteroseismology**

Following the linear theory of stellar oscillations applied to a spherically symmetric equilibrium star, the displacement vector of a nonradial oscillation mode of degree $l$ and azimuthal order $m$ is given by

$$\xi(r, \theta, \phi, t) = [(\xi_l, \xi_r + \xi_{l,0} V_n)] Y^m_l(\theta, \phi) \exp(-i\omega t),$$

with $\theta$ being the angle measured from the polar axis, $\phi$ the longitude, and $\omega$ the angular mode frequency. In the absence of rotation, the modes are called zonal (for which $m = 0$); these reveal $l$ surface nodal lines.

In the presence of rotation, the rotation axis of the star is usually chosen as the axis of the spherical polar coordinate system to describe the modes. In this case, $|m| \leq l$ of the surface nodal lines are lines of longitude. A distinction is made between prograde $m > 0$ and retrograde $m < 0$ modes, corresponding with waves traveling with and against the rotation, respectively. The Coriolis acceleration due to the rotation of the star causes rotational splitting of the mode frequencies into $2l + 1$ components called multiplets. The level of splitting among the multiplet components is determined by the rotation frequency $\Omega$ of the star, where larger splitting is due to faster rotation.

The radial order of the mode, $n$, representing the number of nodes of $\xi_r$, cannot be deduced from surface observables. Rather, it can be estimated from comparison between observed and theoretically predicted frequencies after identification of $(l, m)$. We refer to the monograph Aerts et al. (2010) for detailed methodology to achieve this.
2.2. Asteroseismic Modeling of Low-Mass Stars

Asteroseismic modeling starts from 1D spherically symmetric equilibrium models based on state-of-the-art input physics. Subsequently, the 3D eigenmodes and scalar eigenfrequencies of these 1D models are computed. This is achieved by perturbing the stellar structure equations defining these 1D equilibrium models, in a linear approach. Comparison of these predicted eigenfrequencies with observed ones then allows improvement of the input physics that went into the 1D models. Such a scheme is iterated upon until agreement between the predicted and observed eigenfrequencies is achieved. For low-mass main-sequence stars, such modeling is based on damped p modes excited by the envelope convection, which reveal themselves with their individual frequencies superposed on a power excess due to granulation (e.g., Kallinger et al. 2014). The p modes have short periods of a few minutes and mainly probe the outer envelope. Given the slow rotation of such stars, their zonal \((m = 0)\) p modes can be exploited while neglecting the Coriolis and centrifugal forces. Also, the Lorentz force has a negligible effect on the oscillations compared to the pressure force and gravity. The asteroseismic modeling of the zonal-mode frequencies is often based on the same input physics as for the 1D solar model derived from helioseismology (Christensen-Dalsgaard 2002).

Low-degree zonal p modes in low-mass stars reveal a characteristic frequency spacing related to the mean density of the star, whereas the frequency at maximum power is related to the acoustic cutoff frequency depending on \(M\), \(R\), and \(T_{\text{eff}}\). A spectroscopic measurement of \(T_{\text{eff}}\) thus allows derivation of \(M\) and \(R\) with respect to those of the Sun. This potential was predicted long before the space era of asteroseismology in the seminal paper by Kjeldsen & Bedding (1995) and has since been applied to hundreds of low-mass dwarfs and subgiants (Chaplin et al. 2014). Once \(M\) and \(R\) have been derived, a model-dependent stellar age follows. This procedure leads to relative precisions of \(\sim 2\%\) in radius, \(\sim 4\%\) in mass, and \(\sim 10\%\) in age for stars that were monitored during the four-year nominal Kepler mission and that have similar metallicity and interior physics as that of the Sun (Silva Aguirre et al. 2017).

Evolved low- and intermediate-mass stars offer, in addition to their p modes, the opportunity to exploit mixed dipole \((l = 1)\) modes. These have a p-mode character in the envelope and a g-mode character in the deep interior. In contrast to p modes, g modes have characteristic period spacings determined by the buoyancy frequency \(N(r)\) (cf. Supplemental Figures 1–4). Such spacings were detected from ground-based photometry for white dwarfs long before space asteroseismology (see, e.g., Kawaler et al. 1999, for an overview). Red giants reveal mixed modes with gravity-dominated or pressure-dominated character. Evolved stars thus offer both frequency spacings from their \(l = 0, 2\) p modes and period spacings from their gravity-dominated mixed \(l = 1\) modes. The potential of mixed modes was studied theoretically by Dupret et al. (2009) and discovered in Kepler data by Beck et al. (2011) and Bedding et al. (2011). The overall probing power of mixed dipole modes allows one to pinpoint not only the nuclear burning phase of red giants—an assessment that cannot be done from surface measurements (Bedding et al. 2011, Mosser et al. 2014)—but also how the star’s core rotates (Beck et al. 2012, Mosser et al. 2012, Deheuvels et al. 2014; cf. Section 2.4). Extensive reviews on 1D red giant modeling based on frequency and period spacings of their damped modes are available from Chaplin & Miglio (2013) and Hekker & Christensen-Dalsgaard (2017), to which we refer for details.

2.3. Asteroseismic Modeling of Intermediate- and High-Mass Stars

The overall mixing profile beyond the Schwarzschild boundary of the convective core in intermediate-mass and high-mass stars, \(D_{\text{mix}}(r > r_{\text{cc}})\), is affected by several phenomena, such as convective overshooting/penetration, semiconvection, rotationally and magnetically induced
instabilities, thermohaline mixing, etc. Theoretically predicted instabilities in the radiatively stratified envelope due to rotation or magnetism lead to discontinuities in $D_{\text{mix}}(r > r_{\text{cc}})$. These often remain invisible in evolutionary tracks, but they affect 3D oscillation mode computations. The latter offer a powerful way to calibrate the (free parameters assigned to) mixing and angular momentum transport due to instabilities. In order to achieve that, asteroseismic modeling of an observed star is best done by relying on 1D nonrotating nonmagnetic equilibrium models and by computing their mode frequencies. The computation of these 3D oscillations must consider the accelerations due to rotation in the perturbed equation of motion. This allows us to select the best equilibrium model for the stellar interior and to estimate its free parameters, such as mass, metallicity, and age. Deviations between the predicted and measured oscillation frequencies then reveal shortcomings in the input physics of the equilibrium models and offer a guide to improve it up to the level of precision of the identified oscillation frequencies.

Early asteroseismic modeling of the high-mass B3V star HD 129929 from six identified low-order $l = 0, 1$, and 2 modes detected in ground-based data revealed core convective penetration with $\alpha_{\text{ov}} = 0.15$ (Aerts et al. 2003, Dupret et al. 2004). In addition, the rotationally split modes led to $\Omega_{\text{core}}/\Omega_{\text{ov}} \sim 4$. Following this initial study, major ground-based multisite campaigns lasting many months and involving tens of astronomers were organized [see Handler et al. (2006) and Briquet et al. (2012) for some examples]. These led to $\alpha_{\text{ov}} \in [0, 0.45]$ $\Omega_{\text{p}}$, assuming convective penetration [see, e.g., Aerts (2015) for a summary]. So far, space missions have observed few high-mass stars and none with suitable identified modes to improve asteroseismic modeling achieved from the ground.

Intermediate-mass stars offer better opportunities, as they reveal tens of low-degree, high-order $g$ modes, with periods of half to a few days. The first detection of $g$-mode period spacings in main-sequence stars came from the CoRoT mission (Degroote et al. 2010) and revealed optimal probing power in the near-core region of stars with a convective core. Such $g$-mode spacings have meanwhile been detected for a whole range of rotation rates, from very slow to 80% critical (cf. Section 2.4). Because the $g$-mode periods may be of similar order to the rotation period, typically 1 to 3 d, the Coriolis force is a mandatory ingredient in the 3D oscillation computations to interpret the observed period spacing patterns of $g$ modes, as demonstrated by Van Reeth et al. (2015, 2016) and Ouazzani et al. (2017) for F-type pulsators and by Moravveji et al. (2016), Pápics et al. (2017), Saio et al. (2018b), and Szewczuk & Daszyńska-Daszkiewicz (2018) for B-type pulsators. Unlike for low-mass stars, asteroseismic modeling based on $g$ modes in main-sequence stars therefore starts with estimation of the near-core rotation rate $\Omega_{\text{core}}$, as explained in Section 2.4.

Another complication compared with low-mass stars is the need to include physical ingredients that do not occur in the Sun, such as convective overshooting/penetration from the core instead of the envelope. Furthermore, oscillation modes may be subject to mode trapping in the near-core region. This terminology is used to indicate that changes in the eigenfunctions of the modes may occur owing to sharp features in $N(r)$. As outlined by Miglio et al. (2008), the shape of the composition gradient in the zone left behind by the receding convective core affects the term $\nabla \mu$ in Equation 4, and it thus determines the occurrence and efficiency of mode trapping. The interplay between the size and evolution of the convective core and its generated $\mu$ gradient, on the one hand, and the properties of the near-core mixing captured in $D_{\text{mix}}(r > r_{\text{cc}})$, on the other hand, strongly affected the mode trapping, and this provides a way to age stars (cf. Schmid & Aerts 2016).

Aerts et al. (2018) and Johnston et al. (2019) developed a methodological framework to perform asteroseismic modeling of single and binary stars with a convective core, based on tens of identified $g$ modes of consecutive radial order ($\mu$ typically between $-5$ to $-50$), to estimate $M, X, Z, \Omega_{\text{core}}$, etc. These led to $\alpha_{\text{ov}} \in [0, 0.45]$ $\Omega_{\text{p}}$, assuming convective penetration [see, e.g., Aerts (2015) for a summary]. So far, space missions have observed few high-mass stars and none with suitable identified modes to improve asteroseismic modeling achieved from the ground.

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D_{\text{mix}}(r > r_{\text{co}}), and X_{x}, where X and Z are the relative mass fractions of hydrogen and metals such that X + Y + Z = 1 (with Y being the helium mass fraction), and X_{x} is the hydrogen mass fraction in the convective core (a proxy of the age during the main sequence). Their method adopts the so-called traditional approximation of rotation (TAR) for the treatment of the Coriolis force (e.g., Townsend 2003a,b). Estimation of $D_{\text{mix}}(r > r_{\text{co}})$ by g modes is particularly powerful in the core overshoot zone and allows one to derive the core mass of the star (Moravveji et al. 2015, 2016). Furthermore, the modeling required that $D_{\text{mix}}(r > r_{\text{co}}) \neq 0$ in order to explain the observed g-mode trapping properties, with measured levels of $D_{\text{mix}} \in [1, 1000] \, \text{cm}^2 \, \text{s}^{-1}$. This is smaller than predictions by the theory of rotationally or magnetically induced instabilities and provides a new way to calibrate and guide that theory (e.g., Mathis 2013) and numerical simulations under stellar conditions (Rogers & McElwaine 2017, Pedersen et al. 2018).

Element and angular momentum transport in stars are intimately related because they result from the same processes. Often, they are decoupled from each other in stellar evolution codes, in which each of the transport phenomena is given its own free parameter because of a lack of transport theories based on first principles. From here on, we focus on angular momentum transport, given that the asteroseismic measurement of $\Omega_{\text{core}}$ is quasi independent of 1D equilibrium models, whereas estimation of the mixing profile inside the star does depend on it (cf. Table 1).

We refer to Salaris & Cassisi (2017) for a review of the theory of element transport in stars prior to the asteroseismic estimates available today.

2.4. Asteroseismic Derivation of Interior Rotation

The derivation of the interior rotation rates from asteroseismology cannot be done for all pulsators, because it requires the detection and identification of specific oscillation modes. Detection of rotationally split, mixed, or g modes is required to derive $\Omega_{\text{core}}$, whereas split p modes lead to $\Omega_{\text{env}}$. The approach to follow depends on the ratio of $\Omega$ and $\omega$ and is, therefore, different for p and g modes in different types of stars. Whenever $\Omega/\omega \lesssim 25\%$, one can rely on a perturbative approach to assess the effects of the Coriolis and centrifugal forces on the stellar oscillations. This is typically the case for p modes in most stars, p and mixed modes in red giants, and g modes in subdwarfs and white dwarfs. By contrast, $\Omega/\omega \gtrsim 25\%$ requires a nonperturbative treatment of the Coriolis and/or centrifugal forces. Regarding the Coriolis force, this is conveniently done by the TAR for g modes in intermediate- and high-mass stars in the case in which $2\Omega \ll N$.

2.4.1. From a perturbative approach. Both the Coriolis and centrifugal forces change the mode frequencies. Following a first-order perturbative approach in the computation of the oscillation frequencies is fine when the centrifugal force ($\sim \Omega^2$) and, hence, the deformation of the star can be ignored and when $\Omega \ll \omega$. Let us assume that this is the case and that the rotation rate only changes with depth $r$ inside the star (and not with latitude). Under these assumptions, the mode frequency $\omega_{nl}$ belonging to eigenvector $\xi_{nl} = (\xi_{r, nl}, \xi_{h, nl})$ in the nonrotating case gets split into $2l + 1$ frequency multiplet components due to the Coriolis acceleration. Furthermore, each of those components gets shifted due to the Doppler effect according to the rotation between a coordinate system corotating with the star and the inertial coordinate system of the observer. For example, in the simplest case of uniform rotation, the transformation between the corotating frame and the observer’s inertial frame implies a frequency shift of $m\Omega$. For a general rotation law $\Omega(r)$, the frequencies of the multiplet in the inertial frame become

$$\omega_{nlm} = \omega_{nl} + m (1 - C_{nl}) \int_{0}^{R} K_{nl}(r) \Omega(r) dr,$$

where $D_{\text{mix}}(r > r_{\text{co}})$, and $X_{x}$, where X and Z are the relative mass fractions of hydrogen and metals such that X + Y + Z = 1 (with Y being the helium mass fraction), and $X_{x}$ is the hydrogen mass fraction in the convective core (a proxy of the age during the main sequence). Their method adopts the so-called traditional approximation of rotation (TAR) for the treatment of the Coriolis force (e.g., Townsend 2003a,b). Estimation of $D_{\text{mix}}(r > r_{\text{co}})$ by g modes is particularly powerful in the core overshoot zone and allows one to derive the core mass of the star (Moravveji et al. 2015, 2016). Furthermore, the modeling required that $D_{\text{mix}}(r > r_{\text{co}}) \neq 0$ in order to explain the observed g-mode trapping properties, with measured levels of $D_{\text{mix}} \in [1, 1000] \, \text{cm}^2 \, \text{s}^{-1}$. This is smaller than predictions by the theory of rotationally or magnetically induced instabilities and provides a new way to calibrate and guide that theory (e.g., Mathis 2013) and numerical simulations under stellar conditions (Rogers & McElwaine 2017, Pedersen et al. 2018).

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$$\omega_{nlm} = \omega_{nl} + m (1 - C_{nl}) \int_{0}^{R} K_{nl}(r) \Omega(r) dr,$$
where

\[ K_{nl}(r) = \frac{(\xi^l_0)^2 + [(l+1)](\xi^l_{-1} - 2\xi^l_1 - \xi^l_{-1}) r^2 \rho}{\int_0^R (\xi^l_0)^2 + [(l+1)](\xi^l_{-1} - 2\xi^l_1 - \xi^l_{-1}) r^2 \rho dr} \]

is called the unimodal rotational kernel and

\[ C_{nl} = \frac{\int_0^R (\xi^l_{-1} + \xi^l_{+1}) r^2 \rho dr}{\int_0^R (\xi^l_0)^2 + [(l+1)](\xi^l_{-1} - 2\xi^l_1 - \xi^l_{-1}) r^2 \rho dr} \]

is the so-called Ledoux constant (Ledoux 1951). Both \( C_{nl} \) and \( K_{nl}(r) \) depend on the 1D equilibrium model of the star. However, as outlined in detail by Aerts et al. (2010, chapter 3, their figure 3.38), \( C_{nl} \approx 0 \) for high-order or high-degree \( p \) modes. However, the displacement is dominantly horizontal for high-order \( g \) modes and the terms containing \( \xi \) in Equations 7 and 8 can be neglected, leading to

\[ C_{nl} \approx \frac{1}{[(l+1)]}, \]

which is independent of the equilibrium model. In the approximation of uniform rotation, one obtains

\[ \omega_{dln} = \omega_{dnl} + m(1 - C_{nl}) \Omega. \]

In this case the effect of rotation on the oscillation mode frequencies is completely determined by \( C_{nl} \) and does not depend on the rotational kernels. For high-order or high-degree \( p \) modes, for which \( C_{nl} \approx 0 \), the measured rotational splitting between adjacent \( m \) values in a multiplet is then given by the average rotation rate in the stellar envelope, \( \Omega_{\text{env}} \), in a model-independent way.

For low-order \( p \) modes, the derivation of \( \Omega_{\text{env}} \) depends on the 1D equilibrium model through \( C_{nl} \). Depending on the type of star and its evolutionary state, typical values are \( C_{nl} \in [0, 0.5] \), so the derivation of \( \Omega_{\text{env}} \) maximally implies a relative uncertainty of 50% for \( \Omega_{\text{env}} \). Often, however, the model dependence of \( C_{nl} \) is negligible, leading to a high-precision quasi model-independent estimate of \( \Omega_{\text{env}} \). This is the case for the \( \textit{Kepler} \) target KIC 11145123, whose \( p \)-mode splitting is illustrated in Figure 2a. For high-order \( g \) modes, \( C_{nl} \) is fully determined by the mode degree, hence their measured splitting offers a model-independent estimate of \( \Omega_{\text{core}} \). Particularly, the rotational splitting measured from adjacent frequency peaks in a triplet of \( g \) modes equals half the rotation rate. Aside from \( p \) modes, KIC 11145123 also shows rotationally split \( g \) modes. Given the almost equator-on view upon the stellar rotation axis, its dipole \( g \) modes are detected as doublets because the zonal modes almost cancel out in the line of sight. This is illustrated in Figure 2b, where the splitting between the two doublet components offers a direct and model-independent measurement of \( \Omega_{\text{core}} \) (Kurtz et al. 2014).

In the general case of differential rotation, the measured rotational splitting is determined by the eigenvector of the mode, \( \xi_{dnl} \), as well as by the shape of \( \Omega(r) \) and the rotational kernels, which depend on the 1D equilibrium model. In that case, the definition of the corotating frame is not obvious, but a kernel-weighted average of \( \Omega(r) \) is a logical choice to compute the Doppler shift toward an inertial frame. As such, one obtains a kernel-weighted average of \( \Omega_{\text{env}} \) from \( p \)-mode multiplets and a kernel-weighted average of \( \Omega_{\text{core}} \) from \( g \) modes or from mixed-mode multiplets as detected in \( \textit{Kepler} \) data of red giants (Beck et al. 2012, Mosser et al. 2012).

Perturbation theories up to higher order in \( \Omega \) have been developed, e.g., second-order \( p \) modes (Saio 1981); second-order \( p \) and \( g \) modes including magnetism (Dziembowski & Goode 1992);...
Figure 2
Rotational splitting of (a) a quadrupole p mode and (b) a dipole g mode of the intermediate-mass star KIC 11145123. The red dotted line indicates the position of the \((l, m) = (1, 0)\) g-mode frequency. Figure reproduced from public data, after Kurtz et al. (2014).

second-order g modes (Lee & Baraffe 1995); third-order p modes (Soufi et al. 1998, Daszyńska-Daszkiewicz et al. 2002); and second-order p, g, and mixed modes (Suárez et al. 2006). These theories are based on various assumptions regarding the Coriolis, centrifugal, Lorentz, and tidal forces. They are also valid for different regimes of the ratios of the relevant frequencies: \(\Omega, \omega, \omega_A\), and the orbital frequency in the case of a binary. The theories also treat the deformation of the star \(\sim \Omega^2\) differently in the computation of the equilibrium model. Those higher-order perturbation theories lead to splittings and shifts in the frequencies of the oscillations compared with \(\omega_{nl}\) of the nonrotating case. The theories have in common that they lead to coupling of eigenmodes of different \(l\) and \(m\). The period spacing and mode trapping properties of these various perturbation theories may deviate appreciably from those of nonperturbative theories (cf. Saio et al. (2018a), their figure 6, for a concrete comparison between TAR and a second-order theory for g modes).

2.4.2. Based on the traditional approximation. The perturbative approach is no longer appropriate when \(\omega\) is of the same order of, or smaller than, \(\Omega\). The reason is that the Coriolis acceleration is no longer small compared to the acceleration term in \(\omega^2\) in the equation of motion. As outlined by Lee & Saio (1987), the oscillation equations are appreciably simplified when the tangential component of the rotation vector is ignored. This so-called TAR, commonly used in geophysics (Eckart 1960) and in studies of neutron stars (Bildsten et al. 1996), allows modes to be computed from the Laplace tidal equations and offers an excellent approximation for g modes in intermediate- and high-mass main-sequence stars (see, e.g., Townsend 2003a, Mathis 2013, for comprehensive descriptions), even if they rotate up to a large fraction of their critical rate (Ouazzani et al. 2017).
In a nonrotating star with a convective core, the periods of low-degree, high-order g modes have an asymptotic spacing given by

$$\Delta \Pi_l = \Pi_{l,m} - \Pi_{l,m-1} = \frac{\Pi_0}{\sqrt{l(l+1)}}.$$  

where

$$\Pi_0 = 2\pi^2 \left( \int_{r_1}^{r_2} N \frac{dr}{r^4} \right)^{-1},$$

with $r_1$ and $r_2$ being the turning points of $N(r)$, indicative of the boundaries of the convective region(s) that define the g-mode propagation cavity (Tassoul 1980, Smeyers & Van Hoolst 2010). Typical values for $\Delta \Pi_l$ for B-type g-mode pulsators range from 5,000 s to some 12,000 s for $M \in [3, 8] M_\odot$ (Degroote et al. 2010, Moravveji et al. 2015). These values decrease as $l$ increases, according to Equation 11. Van Reeth et al. (2016) computed $\Delta \Pi_l$ for dipole and quadrupole modes of F-type pulsators, varying $M, X, Z, X_c$, and $\omega_{cc}$ and obtained $\Delta \Pi_1 \in [2,500, 3,500]$ s and $\Delta \Pi_2 \in [1,200, 2,200]$ s for nonrotating stars.

Inclusion of the Coriolis force in the TAR, assuming uniform rotation, leads to a similar period spacing pattern:

$$\Delta \Pi_{l,m} = \frac{\Pi_0}{\sqrt{\lambda_{l,m,\varphi}}},$$

with $\lambda_{l,m,\varphi}$ being the eigenvalue of the Laplace tidal equation for the g mode with quantum numbers $l$ and $m$ and $\varphi$ the spin parameter $\varphi \equiv 2\Omega_{core}/\omega_{cc}$. Hence the observed period spacing pattern of a series of g modes with consecutive radial order $n$ and dominant mode numbers $l$ and $m$ allows one to simultaneously identify $l$ and $m$ and estimate the near-core rotation frequency $\Omega_{core}$. The procedure is illustrated in Figure 3 for the gravito-inertial modes detected in the Kepler single stars KIC 7434470 and KIC 11294808 (Van Reeth et al. 2016). From the observed diagnostics, it is found that KIC 7434470 is an $M \simeq 1.4 M_\odot$ star in its early main-sequence evolution with $\Omega_{core}/\Omega_{crit} = 62\%$, whereas KIC 11294808 is an $M \simeq 1.9 M_\odot$ star fairly close to the terminal-age main sequence (TAMS) rotating nearly uniformly ($\Omega_{core}/\Omega_{crit} = 98\%$, Van Reeth et al. 2018) at $59\%$ of its critical rate Mombarg et al. (2019). Their difference in mass and evolutionary stage is not only revealed by the values of $\Delta \Pi_l$ but also by the difference in mode trapping properties: The $\mu$-gradient zone of KIC 11294808 is fairly extensive as its convective core has had a long time to shrink, whereas KIC 7434470 is in the mass range of having a growing convective core early in its evolution and, hence, does not reveal mode trapping. In this way, the period spacing patterns such as those in Figure 3 make it possible to estimate $M, X, Z, X_c$ after derivation of $\Omega_{core}$ (Aerts et al. 2018), as well as put constraints on $D_{aux}(r > r_{cc})$ if mode trapping is observed as it was for KIC 7434470 in Figure 3. The absence of mode trapping, as in the case of KIC 11294808 shown in Figure 3, only allows one to deduce a lower limit of $D_{aux}(r > r_{cc})$. Asteroseismic modeling of gravito-inertial modes has been put into practice for single and binary B- and F-type g-mode pulsators on the basis of four-year Kepler light curves by Moravveji et al. (2015, 2016), Van Reeth et al. (2016), Schmid & Aerts (2016), Guo et al. (2017), Pápics et al. (2017), Ouazzani et al. (2017), Kallinger et al. (2017), and Szewczuk & Daszyńska-Daszkiewicz (2018) at various levels of detail.

Mathis (2009) generalized the TAR to include general differential rotation both in radius and latitude. This allows derivation of an improved expression compared with that in Equation 13 for the period spacing pattern. His formalism was implemented and exploited by Van Reeth et al. (2018) to study the sensitivity of gravito-inertial modes to differential near-core rotation in a
sample of 37 intermediate-mass main-sequence stars (included in Figure 4, discussed below). It was found that differential rotation can only be measured when period spacing patterns for different degrees \( l \) are detected or when the surface rotation rate is measured from rotational modulation or \( p \) modes. The stars for which this is fulfilled all have surface-to-core rotation ratios between 0.95 and 1.05. Prat et al. (2017, 2018) go beyond the TAR by deriving an asymptotic period spacing for axisymmetric subinertial gravito-inertial waves and a general asymptotic theory for gravito-inertial waves propagating within a general differential rotation both in \( r \) and \( \theta \), respectively. These theoretical advances have yet to be applied to interpret data. Also, 2D oscillation codes have been built to compute oscillations of rapidly rotating stars using either realistic 1D spherical stellar models or 2D deformed, often polytropic, models (e.g., Dintrans & Rieutord 2000, Reese et al. 2006, Ballot et al. 2010, Ouazzani et al. 2017).

### 2.4.3. Current status of asteroseismic rotation rates.

Figure 4a shows the asteroseismic estimates of \( \Omega_{\text{core}} \) and \( \log g \) for 1,210 stars available in the literature (d.d. August 1, 2018), based on space photometry that led to suitable identified modes allowing derivation of these two quantities. The rotation rates were derived as illustrated in Figures 2 and 3, whereas the asteroseismic masses and radii were deduced from scaling relations of damped \( p \) modes or from forward modeling of coherent \( g \) modes, as discussed in Sections 2.2 and 2.3. These 1,210 stars cover the entire evolutionary sequence from core-hydrogen burning to the white dwarf remnant phase, for birth masses ranging from 0.72 to 7.9 \( M_\odot \), and rotation rates from essentially zero up to 80% of the critical Roche frequency.\(^1\) Several of the main-sequence stars are in binaries; these have been indicated as such. Although the sample of main-sequence stars is not representative in mass, age, rotation rate, and binarity, the red clump samples leave no doubt that their helium-burning cores

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\(^1\)The data to reproduce Figure 4 is available in electronic format as Supplemental Data 1.
have angular momentum similar to subdwarfs and white dwarfs. This result is found for both red clump stars and secondary red clump stars and is, hence, independent of having undergone a helium flash or not.

Furthermore, Figure 4a reveals that $\Omega_{\text{core}}$ of all red giants is about two orders of magnitude lower than the standard theory of angular momentum transport predicts (e.g., Goupil et al. 2013, Marques et al. 2013, Cantiello et al. 2014, Ouazzani et al. 2018). This means the cores must lose angular momentum to their overlying envelopes much more efficiently than any current theory predicts. Irrespective of their birth masses or binaries, all red giants reveal that their progenitors are subject to yet unknown physical processes that can extract angular momentum from their cores in an efficient manner. It has been suggested that the efficiency of the unknown angular momentum transport process(es) is mass dependent (Eggenberger et al. 2017), but it remains unclear if a dichotomy occurs between stars born with and without a convective core (Tayar & Pinsonneault 2013) or whether the angular momentum transport increases gradually in efficiency with birth mass.

In Figure 4b, we show the 45 stars with a measurement of both $\Omega_{\text{core}}$ and $\Omega_{\text{env}}$ from space photometry. These cover a narrower mass range of $[0.84, 3.25] M_\odot$ but an almost equally broad spread in $\Omega/\Omega_{\text{crit}}$ as Figure 4a. For some of the measurements in Figure 4b, inversion methods like those of helioseismology (e.g., Thompson et al. 2003) have been used to derive $\Omega(r)$ throughout the star, instead of (kernel-)averaged values for the core and envelope. From those inversion studies, the dependence of the results for $\Omega_{\text{core}}$ on the adopted 1D equilibrium models using the rotation kernels has been found to be modest and not important in the context of angular momentum transport theory during stellar evolution [see, e.g., Deheuvels et al. (2012, 2014, 2015), Di Mauro et al. (2016), and Triana et al. (2017) for sub- and red giants and Triana et al. (2015) for a
B-type star]. These inversion studies delivered weighted averages of the rotation in the convective regions and did not lead to radial differential rotation profiles in those zones. Based on the average core and envelope rotation rates, Figure 4 reveals that intermediate-mass main-sequence stars rotate nearly uniformly (unless they are a member of a binary) and that red clump stars are efficient in reducing the level of radial differentiability of the rotation acquired during the subgiant and red giant phases, when they have a radiative core surrounded by a hydrogen-burning shell.

Asteroseismological determination of rotation for main-sequence low-mass stars with damped p modes cannot reveal \( \Omega_{\text{core}} \) but only \( \Omega_{\text{env}} \). By combining their rotationally split p modes with a high-precision spectroscopic measurement of \( v \sin i \) and stellar models or with the frequency of rotational modulation detected in the Kepler data, Benomar et al. (2015) and Nielsen et al. (2017) achieved the envelope and surface rotation rates of 27 such single stars, covering a mass range from 1.0 to 1.6 \( M_\odot \). This revealed nearly uniform radial envelope rotation with average rotation values covering \( \Omega_{\text{env}}/2\pi \in [0.4, 3.2] \muHz \) for the sample, in agreement with the spectroscopic \( \Omega_{\text{surf}} \) values. By contrast, Benomar et al. (2018) studied latitudinal differential rotation in 40 low-mass main-sequence stars in this mass range and found 13 of these to reveal significantly slower polar than equatorial rotation, a phenomenon that also occurs for the Sun (Thompson et al. 2003). Asteroseismic inversions to derive the latitudinal rotation law in the envelope, \( \Omega_{\text{env}}(r, \theta) \), revealed the magnitude of the angular differentiability to be larger than that for the Sun, with latitudinal shear factors between the equator and midlatitude ranging from approximately 0.27 (Benomar et al. 2018, their table S3). Such a level of observed latitudinal shear is much larger than predictions from theory and numerical simulations for the Sun and solar-like stars (Hotta et al. 2015, Brun et al. 2017). Although the 13 stars have faster average envelope rotation than the Sun, with \( \Omega_{\text{env}}/2\pi \in [0.38, 1.7] \muHz \) when adopting a latitudinal rotation law as for the Sun, the asteroseismic inference of these shear values points to large anisotropy in the turbulence, accompanied by efficient angular momentum transport. These asteroseismic results remain unexplained as of 2015. Such a level of shear is not found from the 3D numerical simulations for red giants by Brun & Palacios (2009), who derived antisolar latitudinal differential rotation and strong radial differential rotation in the slowly rotating outer envelope, as it would occur from conservation of specific angular momentum by individual convective flows. Major theoretical uncertainty in the level of radial differentiability of the rotation in convection zones remains for low-mass red giants (e.g., Kissin & Thompson 2015).

Angular momentum transport by IGWs triggered by a convective core in intermediate-mass stars (Rogers et al. 2013) and by the envelope in low-mass stars (Charbonnel & Talon 2005) offers a natural explanation of the observational findings, but there may be other mechanisms as well. In any case, candidate mechanisms must operate and be able to reduce the level of differentiability of \( \Omega \) during both core-hydrogen and core-helium burning for stars with a convective core (Deheuvels et al. 2015, Aerts et al. 2017). Cores of low-mass stars spin up during the subgiant and red giant branch (RGB) phases, to reach a level up to ten times the envelope rate, irrespective of whether they belong to a binary or not.

Figure 4 shows \( \Omega_{\text{core}} \) rather than the angular momentum of the stellar core, \( J_{\text{core}} \equiv M_{\text{core}} \cdot \Omega_{\text{core}} \cdot R_{\text{core}}^2 \), with \( M_{\text{core}} \) and \( R_{\text{core}} \) being the mass and radius of the stellar core, respectively. The reason is that the asteroseismically derived \( \Omega_{\text{core}} \) is (quasi-)model independent, whereas the computation of \( J_{\text{core}} \) requires knowledge of the core mass and core radius, which are model dependent. A discrepancy of two orders of magnitude between the observed angular momentum of young neutron stars and white dwarfs, on the one hand, and the theoretical predictions for \( J_{\text{core}} \) of their progenitor stars, on the other hand, was previously reported (see Langer 2012, his figure 1). Asteroseismology has now made it clear that this mismatch must be sought in the early phases of stellar evolution, at least for low- and intermediate-mass stars. Their cores lose far more angular
### SUMMARY POINTS FROM OBSERVATIONS

1. Asteroseismology makes it possible to determine the interior rotation of stars, provided that they offer suitable nonradial oscillations to do so; in the cases of rotational splitting detected for mixed or high-order g modes, this is achieved in a model-independent way.

2. *Kepler/K2* space photometry of micromagnitude precision and four-year duration led to the core rotation rates of more than 1,200 stars covering the mass range $[0.7, 8] \ M_\odot$, rotation frequencies up to 80% of the critical Roche frequency, and evolutionary phases from core-hydrogen burning to the white dwarf remnant phase.

3. Low- and intermediate-mass main-sequence stars with an asteroseismic measurement of the core rotation rate have nearly uniform radial rotation.

4. Latitudinal envelope rotation measured in 13 low-mass main-sequence stars revealed larger shear and anisotropy than those of the Sun.

5. Stars lose a large fraction of their initial core angular momentum between the core-hydrogen burning phase and the end of core-helium burning.

6. The asteroseismically measured angular momentum of the core of single low- and intermediate-mass stars in the core-helium burning phase agrees with the angular momentum of subdwarfs and white dwarfs.

7. Current theory of angular momentum transport falls short by one-to-two orders of magnitude to explain the asteroseismic core rotation rates of evolved low- and intermediate-mass stars.

momentum than predicted by theory, so an efficient mechanism is required to transport it from the core to the outer envelope. Despite the limited sample of main-sequence stars in Figure 4, it is found that the angular momentum transport happens efficiently when the star has a convective core. High-precision asteroseismic measurements of both $\Omega_{\text{env}}$ and $\Omega_{\text{core}}$ from space photometry are currently lacking for high-mass stars and, in particular, for blue supergiants. As discussed in Section 4, this will be remedied in the coming years. See the sidebar titled Summary Points From Observations.

Armed with these recent asteroseismic constraints on $\Omega(r, \theta)$, we now turn our attention to new developments in the theory and simulation of angular momentum transport, offering the reader a historical overview of the relevant facets of this topic and stressing the need for improvements.

### 3. ANGULAR MOMENTUM TRANSPORT

In the standard treatment of stellar structure and evolution, radiative zones are assumed to be motionless. However, as proven by helio- and asteroseismology and by observed surface abundances, they are the seed of dynamical processes, which act on secular timescales to transport angular momentum and mix chemical elements. Here, we focus only on angular momentum transport. Four main processes are identified (Mathis 2013, Zahn 2013): meridional circulation, turbulence driven by instabilities, magnetism, and internal waves. Together these mechanisms give rise to angular momentum transport within radiative interiors of stars. Note, however, that the term rotational transport/mixing is sometimes used by the stellar evolution community as terminology for these processes altogether.

#### 3.1. Hydrodynamical Meridional Circulation: From Eddington–Sweet to Zahn’s Model

Historically, the large-scale meridional circulation, which stirs stellar radiative zones, was ascribed to the deformation of the isobars by the centrifugal acceleration, the Lorentz force if there is a...
Baroclinic state: a state when surfaces of constant pressure and temperature do not coincide.

magnetic field, and the tidal force if there is a close companion. In that case, the radiative flux is no longer divergence free (von Zeipel 1924) and should be balanced by heat advection. In the treatment derived by Eddington (1925) and Sweet (1950), the meridional circulation velocity was thus linked to the ratio of the centrifugal acceleration (and the other perturbing forces) to the gravity. The characteristic timescale of this flow was derived by Sweet (1950) and named the Eddington–Sweet timescale: 

\[ t_{ES} = \frac{t_{KH}}{\Omega_1^2 R^3}, \]

where \( t_{KH} = \frac{GM^2}{RL} \) is the Kelvin–Helmholtz timescale. This timescale is longer than the nuclear timescale representing the main-sequence evolution. These results predicted that rapid rotators should be well mixed because of this circulation and should modify their evolution to the giant branch compared with nonrotating stars; this is not observed. This was improved upon by Mestel (1953), who accounted for the mean molecular weight (i.e., the \( \mu \)-gradients), which reduces these circulation effects.

However, the fact that meridional circulation advects angular momentum was ignored. After a first transient phase lasting an Eddington–Sweet time, the star settles into an asymptotic regime. At this stage, the circulation is driven by structural adjustments, torques apply to the star, and when we restrict treatment to the simplest case ignoring magnetic fields and internal waves, internal stresses such as those related to shear-induced turbulence occur (Zahn 1992, Rieutord 2006). On one hand, if the star loses angular momentum through a wind, the circulation adjusts to transport angular momentum toward the surface (Zahn 1992, Maeder & Zahn 1998, Mathis & Zahn 2004, Decressin et al. 2009). In that case, the characteristic timescale of the circulation is that of the braking of the star by its wind. The induced rotation resulting from the advection of the angular momentum is then nonuniform and a baroclinic state sets in, where the temperature varies with latitude along the isobar. On the other hand, if the star does not exchange angular momentum with its environment, the advection by the circulation balances the internal stresses (this is the so-called gyroscopic pumping). In that case, the characteristic timescale of the circulation is the one connected with the more efficient stress that transports angular momentum (e.g., turbulence, waves, or magnetic fields). In the case of a uniform rotation without any turbulent transport, magnetism, and waves, the circulation thus dies out (Busse 1982). In the case of late-type stars, it is hence the loss or gain of angular momentum that drives the circulation rather than the amplitude of the angular velocity or the related centrifugal acceleration.

The loop of angular momentum transport in stellar radiative zones by large-scale circulation can be identified as follows Rieutord (2006, Decressin et al. 2009): First, meridional currents are sustained by torques applied at the stellar surface, internal stresses such as the viscous ones related to turbulence, and structural adjustments; next, the temperature relaxes to balance the advection of entropy by the meridional circulation; finally, because of the baroclinic torque induced by the latitudinal distribution of temperature fluctuations on the isobar, a new differential rotation profile is established through the so-called thermal-wind balance. This shear may again generate turbulence, closing the loop.

3.2. Hydrodynamical Instabilities

Differential rotation in convectively stable radiative zones can induce a large diversity of hydro- and magnetohydrodynamical instabilities, as discussed below. The turbulence induced by these instabilities can then transport angular momentum. It was already stressed above that mixing at convective-radiative interfaces is critical for both angular momentum and chemical evolution.

The hydrodynamical instabilities taking place in the bulk of stellar radiative zones are classified in three main families: the instability related to the Rayleigh criterion for differential rotation, baroclinic instabilities, and shear instabilities. The two first families draw their energy from the
potential energy built up by the entropy stratification and the centrifugal acceleration, whereas the shear instabilities’ energy originates from the kinetic energy of the medium.

3.2.1. The Rayleigh criterion for the stability of differential rotation. The first major instability for a differentially rotating fluid is directly linked to the Coriolis acceleration. It was first studied by Lord Rayleigh (1916) in the case of a homogeneous and inviscid flow and by Taylor (1923) in the case of a viscous fluid.

Let $s$ be the distance to the axis of rotation in cylindrical coordinates. The so-called Rayleigh or epicyclic frequency $N_\Omega$ is defined as

$$N_\Omega^2 = \frac{1}{s^3} \frac{d}{ds} \left[ (s^2 \Omega)^2 \right].$$

The differently rotating fluid is stable if $N_\Omega^2 > 0$ and unstable if $N_\Omega^2 < 0$. The Rayleigh frequency corresponds to the response time to the restoring Coriolis acceleration and therefore plays an analogous role to the Brunt–Väisälä frequency ($N$) when considering fluctuations in the entropy and chemical stratification, as already discussed from Equation 4. Following Solberg (1936) and Høiland (1941), the Rayleigh criterion can be generalized to take into account the stratification. A differentially rotating stratified region is stable if $N^2 + N_\Omega^2 > 0$ and if the specific angular momentum per unit mass, $j = s\Omega$, grows from the pole toward the equator on each isentropic surface.

3.2.2. Baroclinic instabilities. In the case of a general differential rotation law rather than uniform or cylindrical rotation ($\Omega(s)$), stars are in a baroclinic situation in which the entropy gradient and the gravity are not aligned (e.g., Zahn 1992). This configuration can lead to unstable axisymmetric displacements. The growth of these displacements is hindered by the stable stratification, but this stabilizing effect gets reduced on small scales by thermal diffusivity. Therefore, these instabilities need to be described in the presence of thermal diffusivity $K_T$ and viscosity $\nu$, keeping in mind that $K_T$ is several orders of magnitude larger than $\nu$. In a stellar radiative region with an unstable differential rotation and a stable entropy stratification, the effect of heat and momentum diffusion is to weaken the effect of stratification by the ratio $\nu/K_T$. Such a situation in which heat is diffused faster than momentum leads to a so-called double diffusive instability like semiconvection if $N_T^2 < 0$ and $N_\mu^2 > 0$ and to thermohaline instability if $N_T^2 > 0$ and $N_\mu^2 < 0$. Goldreich & Schubert (1967) and Fricke (1968) identified this instability, which is now known as the Goldreich–Schubert–Fricke (GSF) instability, and derived the instability criteria:

$$\frac{\nu}{K_T} N_T^2 + N_\Omega^2 < 0 \quad \text{or} \quad |\partial_z \Omega|^2 > \frac{\nu}{K_T} N_T^2,$$

where $z$ is the coordinate along the rotation axis. The first criterion is the Solberg–Høiland condition modified by the presence of dissipative processes, whereas the second one is directly linked to the baroclinicity of the star when $\partial_z \Omega \neq 0$. These criteria can be generalized in the case in which the chemical stratification is considered. The first instability criterion then becomes

$$\frac{\nu}{K_T} N_T^2 + \frac{\nu}{K_\mu} N_\mu^2 + N_\Omega^2 < 0,$$

where $K_\mu$ is the molecular diffusivity, which is generally of the same order of magnitude as the viscosity. In the case of a strong stabilizing chemical stratification, the GSF instability can be inhibited. However, another axisymmetric instability, the axisymmetric-baroclinic-diffusive instability is triggered because of the action of heat diffusion (Knobloch & Spruit 1983). The instability
criterion then becomes
\[ \frac{v}{K_T} (N_T^2 + N_\mu^2) + N_\Omega^2 < 0. \]
Finally, nonaxisymmetric baroclinic instability can be triggered if
\[ \frac{|\partial \ln \Omega}{\partial \ln r} > C (\frac{N}{\Omega})^2 H_0 \min (H_\rho, H_N, H_\Omega) \]
where \( H_X = |d \ln X/dr|^{-1} \), with \( X = \{\rho, N, \Omega\} \) while \( C \) is a coefficient close to unity (Spruit et al. 1983, Zahn 1983). This development is favored in weakly stratified regions such as layers close to convective/radiative interfaces.

All these instabilities may play a role in the transport of angular momentum and the mixing of chemicals in stellar interiors (Hirschi & Maeder 2010). Usually they are implemented as a diffusion coefficient in 1D stellar evolution codes, but their treatment and implementation is challenging because their nonlinear saturation (and the associated characteristic timescale) is neither properly understood nor modeled.

### 3.2.3. Shear instability

The second family of hydrodynamical instabilities in stellar radiative zones concerns those triggered by vertical and horizontal shears (Zahn 1992). In a nonstratified case, the necessary condition to have an instability is that the velocity profile \( V \) has an inflection point; i.e., \( d^2 V/dx^2 = 0 \), where \( x \) is the direction along which the instability is considered. In the case of a stably stratified differentially rotating region, the situation becomes more complex (Zahn 1992, Mathis et al. 2018). Indeed, the competition between the destabilizing action of the shear and the stabilizing buoyancy and Coriolis forces has to be examined in detail.

If we first consider the effect of a vertical shear (i.e., along the entropy stratification), assuming as a first step that \( 2\Omega \ll N \), the instability is obtained in an adiabatic configuration if \( Ri = N^2/S^2 < 1/4 \), where we have introduced the Richardson number (\( Ri \)) and the shear \( S = dV/dz \). This situation is not reached generally in stellar interiors. However, as in the case of the GSF instability, the action of heat and momentum diffusion should be taken into account. More specifically, because heat is diffused more rapidly than momentum, the heat diffusion weakens the stabilizing action of the entropy stratification. The instability criterion then becomes (e.g., Zahn 1992)
\[ \frac{N^2 v \ell}{S^2 K_T} < Ri_c, \]
where \( Ri_c \) is the critical Richardson number while \( v \) and \( \ell \) are characteristic turbulent velocities and length scales, respectively. This leads to the related turbulent transport coefficient along the vertical direction,
\[ D_{V,V} = \frac{Ri_c}{3} K \left( \frac{r \sin \theta \ d\Omega}{N^2 dr} \right)^2, \]
which has been validated by recent high-resolution direct numerical simulations in a Cartesian geometry (e.g., Prat & Lignières 2013). Let \( D_{ij} \) designate the eddy diffusivity corresponding to the turbulent transport along direction \( i \), because of the instability of the shear in the direction \( j \). In this notation, the characteristic timescale is given by \( R^2/D_{ij} \). The turbulent motions sustained by the instability of the vertical shear are three-dimensional. Consequently, they also trigger turbulent transport along the horizontal direction orthogonal to the entropy stratification (Mathis et al. 2018). In stably stratified rotating fluids, the turbulent transport is anisotropic. Indeed, the buoyancy inhibits turbulent motions in the vertical direction, whereas the Coriolis acceleration
acts as the restoring force along the horizontal direction. In stellar interiors, where $2\Omega \ll N$, this leads to the horizontal turbulent transport coefficient derived by Mathis et al. (2018):

$$D_{H,V} = \frac{N^4 \tau^2}{2\Omega^2} D_{V,V} \quad \text{with generally} \quad D_{H,V} \gg D_{V,V},$$

where $\tau$ is a characteristic turbulent timescale that depends on the vertical shear and rotation. We refer to Mathis et al. (2018) for a detailed discussion of this timescale.

Finally, a horizontal shear [i.e., $\Omega(\theta)$] can trigger finite-amplitude instabilities (Zahn 1983, Richard & Zahn 1999, Dubrulle et al. 2005). This was the first source of turbulence invoked to lead to efficient horizontal transport of momentum and forms the corner stone of the shellular rotation approximation derived by Zahn (1992). Few and incomplete prescriptions have been derived as of today for the related horizontal turbulent transport coefficient $D_{H,H}$. The first one, based on dimensional arguments, was proposed by Zahn (1992). However, his prescription led to configurations that do not respect the condition $D_{V,V} \ll D_{H,H}$ along stellar evolution Maeder (2003). Subsequently, Maeder (2003) and Mathis et al. (2004) derived two other prescriptions. The first one was based on energetic considerations about the horizontal shear while the second one relied on results obtained for turbulent transport in a nonstratified Couette–Taylor experiment (Richard & Zahn 1999, Dubrulle et al. 2005). Both prescriptions lead to larger values for $D_{H,H}$ with $D_{V,V} \ll D_{H,H}$, but new theoretical, numerical, and experimental efforts should be pursued to provide a robust ab initio prediction.

### 3.2.4. Application of the hydrodynamical processes to stars.

Secular 1D stellar modeling has shown that the combined action of hydrodynamical turbulent transport and the advection by the meridional circulation can explain some observed properties of massive stars (e.g., Meynet & Maeder 2000). Theoretical isochrones based on these processes have better capacity to explain observed ones. Furthermore, the induced mixing can reproduce enhancements of helium and nitrogen observed at the surface of some stars (e.g., Hunter et al. 2008). These processes also allow us to reproduce the observed proportion of blue to red supergiants when combined with a suitable prescription for the mass loss. However, as explained in Section 1.2, these successes only apply to a fraction of massive stars. Furthermore, these phenomena explain neither the observed rotation profile of the solar radiative interior (e.g., Pinsonneault et al. 1989, Turck-Chièze et al. 2010, Mathis et al. 2018) nor the level of differential rotation of low- and intermediate-mass stars shown in Figure 4. Finally, challenges remain in reproducing the observed properties of the mixing of light elements, such as lithium. Although the correct rate for lithium is found from recent cluster analyses (Cummings & Kalirai 2018), the observed lithium abundance is not different in single and tidally locked binary stars as expected, given that the loss of angular momentum that drives the meridional circulation is balanced by the action of tides in these objects (Zahn 1994). Clearly, additional physical mechanisms of angular momentum must be sought to explain all observations. Magnetic fields and IGWs are obvious candidates.

### 3.3. Angular Momentum Transport by Magnetism

As already highlighted in Section 1.2, the magnetic fields of low-mass stars are highly variable dynamo fields, whereas those observed in about 10% of the intermediate- and high-mass stars are strong stable structured large-scale fossil fields. A dynamo-generated field in the convective core of main-sequence stars would not penetrate the extended overlying radiative zone (MacGregor & Cassinelli 2003). The typical decay time of a fossil magnetic field is given by $\tau_D \approx R^2/\eta$, where $\eta$
is the magnetic diffusivity. This is approximately $10^9$ to $10^{10}$ years, so a fossil magnetic field would be present throughout the stars’ evolution, if acted on solely by nonturbulent resistivity.

Although the nature and cause of magnetism in stars varies, we are concerned with the angular momentum transport mediated by it, rather than with its origin, so we do not discuss the generation mechanism. Furthermore, because angular momentum transport within convection zones is efficient due to the turbulent motions (magnetic or otherwise), we focus on angular momentum transport by magnetic fields within radiative regions.

Probably the simplest form of angular momentum transport by magnetic fields comes from the work of Ferraro (1937). Let us start with the magnetic induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},$$  \hspace{1cm} (22)

assuming that the magnetic diffusivity $\eta$ is constant. We consider a rotating star with rotation rate $\Omega$ and with a magnetic field $\mathbf{B}$ that is symmetric about the rotation axis. We decompose the magnetic field into poloidal and toroidal components, such that $\mathbf{B} = \mathbf{B}_p + \mathbf{B}_t$, where $\mathbf{B}_p \cdot \hat{\phi} = 0$ and $\mathbf{B}_t = B_\phi \hat{\phi}$. The only velocity component is that in the azimuthal direction, such that $\mathbf{u} = r\Omega \hat{\phi}$, with $\hat{\phi}$ being the unit longitudinal vector. Finally, we neglect magnetic diffusion. The azimuthal component of the magnetic field then evolves according to

$$\frac{\partial B_\phi}{\partial t} = r (\mathbf{B}_p \cdot \nabla) \Omega,$$  \hspace{1cm} (23)

Assuming a steady state, this reduces to $(\mathbf{B}_p \cdot \nabla) \Omega = 0$, which means that $\Omega$ must be constant on poloidal field lines; this is known as Ferraro’s isorotation law. This homogenization of angular velocity along field lines is mediated by Alfvén waves with a speed $v_A = B/\sqrt{\mu_0 \rho}$, where $B$ is the magnetic field amplitude and $\mu_0$ is the permeability of vacuum. In the simplest case, the related characteristic timescale is given by $D/v_A$, where $D$ is the length-scale of variation of the angular velocity along a poloidal field line. Mestel & Weiss (1987) demonstrated it might be very short, even in the case of a weak field ($\approx 10^4$ years for a 1G poloidal field intensity). Different field lines may rotate at different rates. However, if differential rotation occurred, it would produce a magnetic pressure between field lines, which would exchange angular momentum through phase mixing and eventually these variations in rotation would decay, leaving uniform rotation in the presence of a large-scale poloidal field. In the case where the fossil magnetic field connects to adjacent convective zones, some differential rotation can be transmitted to the radiative zone along the field lines (e.g., Strugarek et al. 2011). In the case of strong nonaxisymmetric fields, rotation is uniform (Spruit 1999).

If we do not consider a steady state, we can immediately see from Equation 23 that an initially dipolar field will be wound into a toroidal field with its amplitude increasing in time. The presence of such a toroidal field will reduce the efficiency of the angular momentum transfer because the Alfvén waves that mediate angular momentum transfer are diverted azimuthally. Nevertheless, Mestel & Weiss (1987) found that even a weak magnetic field could impose uniform rotation throughout the star when considering the solar radiative interior as an example. A similarly weak field would be necessary to impose uniform rotation within the radiative zone of a main-sequence star. Of course, the toroidal magnetic field cannot grow indefinitely and is limited by magnetic instabilities.

One expects that the interiors of stars have some degree of radial differential rotation because of the structural adjustments along their evolution and of torques applied at their surfaces. As discussed in Section 1.1, this is due to the continuous spin down of the stellar surface owing to a magnetized wind in the case of low-mass stars and a radiation-driven wind for high-mass stars.
stars, whereas Figure 4 shows that low but significant levels of differentiability are observed for intermediate-mass stars. In such a case, the instabilities that the toroidal components of a magnetic field are likely subject to are the magnetorotational instability (MRI; Chandrasekhar 1960, Balbus & Hawley 1992) and/or the Tayler (1975)/Pitts–Tayler (Pitts & Tayler 1985) instability. These instabilities were reviewed by Spruit (1999) under stellar interior conditions.

The MRI occurs in the presence of a weak field when the angular velocity of the system decreases away from the rotation axis while the angular momentum increases. This instability derives its energy from the differential rotation and has been studied extensively in the astrophysical disks community, where it leads to turbulence within the disk and the requisite angular momentum transfer to allow mass accretion onto the protostar. It has received relatively little attention within the stellar community because the instability is thwarted by the stable stratification. The stability condition in the absence of diffusion is

\[
q = -\frac{\partial \ln \Omega}{\partial \ln r} > \frac{N^2}{2 \Omega^2},
\]

where \(q\) is the differential rotation parameter. For a typical main-sequence star the right hand side is \(\sim 10^5\), requiring an unrealistically strong radial differential rotation for instability to occur. However, at small scales, the stratification can be offset by thermal diffusion, such that the condition becomes

\[
q = -\frac{\partial \ln \Omega}{\partial \ln r} > \frac{N^2}{2 \Omega^2} \frac{\eta}{K_T},
\]

Given that \(\eta/K_T \ll 1\) in stellar interiors, it is not unreasonable to suppose this instability may proceed in some regions of stellar radiative zones at some ages. Arlt et al. (2003) and Jouve et al. (2015) studied this instability in spherical shells, and both found the instability to occur with consequent increased angular momentum transport. However, both of those studies were carried out for an unstratified gas; hence, it is unclear how much those results apply to stellar radiative interiors. The MRI has also been studied within the Sun, where it has been suggested to occur within the tachocline (Kagan & Wheeler 2014).

Spruit (1999) argued that within stellar interiors, the most likely instability is the Tayler instability. This kink-instability derives its energy from the magnetic field itself and is most likely to occur near the poles. The most unstable mode is the \(m = 1\) mode, and the stability condition is

\[
\frac{\partial \ln (B^2 \sin \theta \cos \theta)}{\partial \theta} > 0,
\]

which can often be satisfied in the polar regions. The growth rate of the instability depends on rotation and is proportional to \(\omega_A^2/\Omega\), where \(\omega_A\) is the Alfvén frequency based on the poloidal field strength. Like the MRI, this instability is limited by stable stratification, which can be offset by thermal diffusion at small scales. Accounting for thermal and chemical stratification, the instability criterion considered by Spruit (1999) is

\[
\frac{\omega_A}{\Omega} > \left( \frac{N^2 \eta}{\Omega^2 \kappa} + \frac{N^2}{\Omega^2} \right)^{1/4} \left( \frac{\eta}{\Omega^2} \right)^{1/4},
\]

where \(\kappa\) is the thermal diffusivity. A more complete criterion and its detailed derivation is available from Zahn et al. (2007). Building on the instability criterion in Equation 27, Spruit (2002) claimed that it could sustain a dynamo in the stably stratified regions of stars. In his description, the toroidal field could be generated by differential rotation acting on the initial poloidal field (the standard

MRI: magnetorotational instability
“Ω” effect; see Equation 23). Once this toroidal field is sufficiently strong, it becomes unstable to the Tayler instability. The instability then reproduces the latitudinal and radial field components required to close the dynamo loop.

Angular momentum transport by magnetic fields must ultimately be derived from the Lorentz (Laplace) force in the azimuthal component of the momentum equation (Charbonneau & MacGregor 1993):

\[
\rho r^2 \sin^2 \theta \frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0} B_\phi \cdot \nabla (r \sin \theta B_\theta) + \nabla \cdot (\rho \nu r^2 \sin^2 \theta \nabla \Omega),
\]

where only magnetic and viscous stresses are considered. If one considers variations in radius to be larger than those in latitude, the Maxwell stresses (or magnetic torques) that arise are \(\sim B_r B_\theta / r\).

Spruit (2002) assumes that the growth of the field saturates when the generation of the field due to the instability is balanced by the diffusion of the field by turbulent magnetic diffusion. The instability field strength is found by considering the largest scale at which the instability can occur despite the restoring force due to buoyancy and the smallest scale at which the instability still grows in the presence of magnetic diffusion. Accounting for the reduction in the buoyancy force at small scales of magnetic diffusion. An estimate for the instability field strength at saturation is then obtained by accounting for the reduction in the buoyancy force due to thermal diffusion when acting at small scales. Coupling this to an estimate for the relationship between azimuthal and radial components of the field, based on the length-scale of the instability, gives an estimate of the Maxwell stress \(B_r B_\theta\), which contributes to angular momentum transport, assuming these fields are maximally correlated. Spruit (2002) goes on to assume that this stress can be written as

\[
S_\Omega \approx \frac{B_r B_\theta}{4\pi} = \rho \nu r \frac{\partial \Omega}{\partial r},
\]

where \(\nu\) is the effective viscosity associated with magnetism. Given the saturation field strengths for \(B_r\) and \(B_\theta\), \(\nu\) can be estimated from the local conditions of the rotation, differential rotation, stratification (both thermal and compositional), and thermal diffusivity. This prescription was implemented into 1D stellar evolution codes (Maeder 2003, Maeder & Meynet 2004, Heger et al. 2005). It was found that this efficient angular momentum transport mechanism could slow down the cores of massive stars sufficiently to be more consistent with observations of pulsar rotation rates. In addition, Eggenberger et al. (2005) demonstrated it can lead to the uniform rotation observed in the solar radiative core. This was taken as evidence that the so-called Tayler–Spruit dynamo exists and transports angular momentum according to the prescription described above and laid out by Spruit (2002). However, Cantiello et al. (2014) demonstrated that it cannot explain the core rotation of red giants shown in Figure 4.

The arguments laid out by Spruit (2002) were conceptually appealing and could explain some lingering observational questions having to do with angular momentum transport. However, there were several physical deficiencies associated with the ansatz, as detailed by Zahn et al. (2007). The most severe deficiencies concern the sustainability and saturation of a Tayler–Spruit dynamo as envisioned by Spruit (2002). Indeed, numerical results by Zahn et al. (2007) indicate that while a Tayler instability does occur, it does not produce a dynamo. Although small-scale radial and latitudinal fields are generated by the instability of the toroidal field, this does not regenerate the large-scale axisymmetric poloidal field, which continuously decays in the simulations by Zahn et al. (2007). Furthermore, rather than the instability being saturated by diffusion, it is saturated by the Lorentz (Laplace) force that reacts back on the differential rotation. Therefore, the amplitudes of the fields estimated by Spruit (2002) should be revised. Finally, rather than angular momentum transport behaving as an enhanced viscosity as in Equation 29, the disturbances to the magnetic
field are propagated as Alfvén waves, which do not enhance viscosity. The numerical results by Zahn et al. (2007) were in conflict with those produced by Braithwaite (2006). Although the latter author claimed to have found a dynamo, his simulations did not include explicit magnetic diffusion and ran less than a magnetic diffusion time, which is insufficiently long. In addition, Braithwaite (2006) permanently sustained differential rotation in his simulations through a body force that does not allow physical back-reaction of the Lorentz force, which generally acts against shear. Recent analytical and numerical work by Goldstein et al. (2018) showed that the occurrence of the Taylor instability may depend on the physical approximation used. We conclude that more work on the Taylor instability in stellar radiative interiors is needed. Despite these limitations of the Spruit ansatz, it remains the defacto magnetic prescription used in many 1D stellar evolution codes and is used widely throughout the stellar evolution community.

3.4. Angular Momentum Transport by Internal Gravity Waves

Studies of IGWs have long since been performed, treating various physical circumstances such as the Earth’s oceans, the atmospheres of planets and moons in the solar system, and stellar atmospheres and stellar interiors. In the context of angular momentum transport in stars, IGWs play an important role as they propagate in the radiation zones of stars. Ratnasingam et al. (2019) recently deduced under which circumstances IGWs become nonlinear for stars of various masses and evolutionary stages. To evaluate the role of IGWs in angular momentum transport, it is necessary to understand the wave generation, propagation, and dissipation for appropriate physical circumstances in stellar interiors.

3.4.1. Wave generation. Angular momentum transport by IGWs depends crucially on the properties of the waves that are generated. Indeed, as discussed in the Section 3.4.2, their propagation and dissipation depend sensitively on their frequency and length-scales. Within stars, propagative IGWs can be generated by any disturbance to the stably stratified region. The most common and efficient mechanisms of generation are an adjacent turbulent convection zone and tidal forcing by a companion star (or planet).

The tidal generation of waves was discussed in detail by Zahn (1975). The companion causes a disturbance at the convective-radiative interface that generates IGWs, which propagate away from the convection zone. Goldreich & Nicholson (1989) considered the case of intermediate- and high-mass main-sequence stars in which the energy of IGWs propagates outward. However, the ansatz is equally valid for stars with convective envelopes, in which case the energy of the IGWs propagates inward. The waves generated are of large scale ($l = 2$ and $m = 2$ if coplanar, $m = 1$ if inclined) and have a frequency equal to the forcing frequency. The dissipation of these waves through critical layers, nonlinear breaking or radiative diffusion (see Section 3.4.2) can lead to efficient angular momentum exchange between the companion’s orbit and the star.

For single stars, the predominant source of propagative IGWs is their generation by the convective core, although, as we pointed out in Section 2, they can also be generated by a convective envelope or by thin convection zones due to local opacity enhancements in radiative envelopes (the $\kappa$-mechanism). The generation mechanism of IGWs has been split into two sources: bulk excitation through Reynolds stresses (e.g., Goldreich & Kumar 1990) and direct excitation through plumes (e.g., Townsend 1966, Schatzman 1993). The formulation by Goldreich & Kumar (1990), which was originally developed for the excitation of $p$ modes, has been the most widely adopted version (Kumar & Quataert 1997, Kumar et al. 1999, Lecoanet & Quataert 2013, Shiode et al. 2013), even though it neglects direct excitation of IGWs by plumes, which is a likely significant source. In that formulation, the generation of IGWs is treated as an inhomogeneous wave
equation with a turbulent source term. Goldreich & Kumar (1990) argued that the source term was dominated by the quadrupole term (Reynolds stresses) because the monopole (nonadiabatic expansion/contraction of fluid) and dipole (buoyancy) terms, though physically larger, virtually cancel each other. Although this assumption and its consequences have been justified in the case of p modes, it is unclear whether it is valid in the case of IGWs. It has been demonstrated by Lecoanet et al. (2015) that treating IGW generation as an inhomogeneous wave equation with a source term does reproduce the results of self-consistent, nonlinear numerical simulations of the generation of IGWs in the context of laboratory experiments. Recent theoretical work and 3D numerical simulations of IGWs triggered by turbulent convection in a Cartesian model by Coustos et al. (2018) predicts the same frequency spectrum for IGWs as in Lecoanet et al. (2015), with a power law $E \propto \omega^{-13/2}$.

In general, the spectra obtained by the theoretical studies based on the Goldreich & Kumar (1990) formulation have a steep power-law dependence on frequency. In this formulation, the convective turnover (or eddy) frequency is generated with large amplitude, whereas higher frequencies are generated with significantly lower amplitudes, with the energy in the waves scaling as $E \propto \omega^{-a}$, with values for $a$ typically between between 3 and 7. This is in stark contrast with numerical results by Rogers & MacGregor (2010), Rogers et al. (2013), Alvan et al. (2014), Augustson et al. (2016), and Edelmann et al. (2017), who find power spectra that are flatter, with $a$ between 1.5 and 3. This difference has a significant impact on the efficiency with which IGWs can transport angular momentum and mix species, whereas the theoretical results from Goldreich & Kumar (1990) lead to less efficient transport and mixing than the numerical results. Although it is unclear why there is such a difference between simulations and theory, there are a few obvious contenders. It is unclear whether the approximations made by Goldreich & Kumar (1990) for p modes are appropriate when applied to IGWs; e.g., the source term lacks plume excitation, the turbulent spectra are assumed to be of Kolmogorov type, and the effects of rotation and magnetism on turbulence are ignored as is the lack of intermittency of convection. In particular, the theoretical work by Goldreich & Kumar (1990) relies heavily on the identification of a dominant turnover frequency within the convection zone, but such a frequency does not occur in numerical simulations.

In the theoretical plume model of excitation (e.g., Schatzman 1993), the waves are generated following a Gaussian function in plume size and incursion time, which results in a wave spectrum that is exponential in frequency and wavelength. This model has been revisited by Pinçon et al. (2017, their figure 3), who showed that this process results in a shallow energy generation at low frequency and a steep one at high frequency. Furthermore, as the plume incursion time is decreased, i.e., as the intrusion becomes more impulsive, the spectrum becomes flatter in frequency, which is in line with numerical simulations. It appears that theoretical plume models match numerical simulations better than those caused by internal stresses (Edelmann et al. 2017), but clearly more efforts should still be undertaken to formulate the turbulent source term theoretically.

Although the numerical simulations include many of the effects neglected in theoretical work, they lack turbulence corresponding to stellar regimes. Furthermore, numerical constraints require enhanced damping of waves. Although there is no way to get to realistic levels of turbulence within the convection zone, the simulations tuned to concrete stellar circumstances explain the asteroseismic observations of main-sequence stars shown in Figure 4 (Rogers 2015). Some of the numerical simulations by Rogers et al. (2013) force the convection stronger than the actual circumstances in stars to compensate for enhanced wave damping. Although this is not ideal, there is no obvious better path forward if the aim is to predict wave amplitudes at the stellar surface.

First attempts to quantify the power density spectra in a homogeneous set of a few tens of intermediate- and high-mass stars observed by CoRoT revealed that granulation cannot explain all the data and led to observed power laws $\propto \omega^{-a}$ with $a \leq 5$ (Bowman et al. 2019). Similar work
performed observational analyses from more precise K2 space photometry for 119 OB-type stars, including blue supergiants (D. Bowman, S. Burssens, M. Pedersen, C. Johnston, C. Aerts, et al., submitted). This led to $a \in [1, 7]$. Before we move on to wave propagation and dissipation, we point out that the combined effects of tidally and convectively induced waves has not yet been considered, a lack that should be remedied in the future.

### 3.4.2. Propagation and dissipation

In considering the propagation of IGWs in stellar interiors, we first discuss propagation of pure IGWs in the absence of rotation and magnetism. In this simplest case, IGWs propagate in stably stratified regions when $\omega < N$. Whenever waves have high frequency $\omega$ close to $N$, the waves set up standing modes caused by internal reflection (Alvan et al. 2015). However, here we focus on low-frequency traveling waves, which propagate throughout the interior and are damped before internal reflection. In the limit $\omega \ll N$, the waves have high radial order, which results in the vertical wavelength ($\lambda_v$) being much smaller than the horizontal one ($\lambda_h$). In that case, the horizontal velocities ($v_h$) are much larger than the vertical velocities ($v_v$), i.e., $\lambda_v/\lambda_h \sim \omega/N \sim v_v/v_h$.

There are three main ways to dissipate IGWs: (a) linear thermal diffusion, (b) occurrence of critical layers, and (c) nonlinear breaking. Thermal diffusion depends sensitively on the wavelength and frequency of the waves, with the damping length in the low-frequency regime $l_\delta \propto K_\delta^{-1} k_\omega^{-3} \omega^3 N^{-1}$. Critical layers occur when the frequency of the wave is the same as the local rotation frequency. Formally, at such layers, the vertical wavelength goes to zero. However, nonlinear effects become important, and the wave effectively breaks, with a damping rate proportional to the square root of the local Richardson number (see Section 3.2.3). Finally, there are two potential sources of nonlinear wave breaking: overturning of the stratification (convective instability) and shear of the wave itself (Kelvin–Helmholtz instability; Thorpe 2018). In stellar interiors, convective instability likely only happens where $N$ approaches 0, such as at the center of low-mass stars and near convective-radiative interfaces. Therefore, in the bulk of stellar interiors, the most common source of nonlinear wave breaking would take the form of a Kelvin–Helmholtz instability (Press 1981). The simplest criterion for this type of breaking is provided by $\varepsilon \equiv v_v / (\lambda_\delta \omega) \sim 1$. In intermediate- and high-mass stars, waves generated at the convective core interface may reach sufficient amplitude to satisfy this criterion.

In the case of rotating stars, waves are modified because of the Doppler effect that creates a systematic frequency difference between prograde and retrograde waves. This leads to thermal damping acting differently on prograde and retrograde waves, thus sustaining a net transport of angular momentum. In the case of rapidly rotating stars, waves with frequencies close to $2\Omega$ are influenced by the Coriolis acceleration. In the case of subinertial waves for which $\omega < 2\Omega$ (see Figure 5), the propagation domain is restricted to the equatorial region. In the case of superinertial waves for which $\omega > 2\Omega$ (see Figure 5), the waves continue to propagate in the full spherical shell as in the nonrotating case because waves are less influenced by the Coriolis acceleration (Lee & Saio 1987, Dintrans & Rieutord 2000). In the case of differential rotation, the behavior of waves is the same but with much more complex propagation cavities than shown schematically in Figure 5 (Prat et al. 2018). This can have important consequences for the transmission of energy from convective to radiative regions and on the transport of angular momentum (Mathis 2009). In the subinertial regime, wave-induced transport of angular momentum is confined to the equatorial region.

Magnetic fields also affect the propagation and dissipation of IGWs in stellar interiors. If the field is strong enough such that $\omega_\lambda \equiv B / (r \sin \theta \sqrt{\mu_0 \rho}) \sim \omega$, IGWs can become trapped (e.g., Rogers & MacGregor 2010, Mathis & de Brye 2011). In the case of a toroidal field at the convective-radiative interface, IGW propagation can be completely blocked; however, in the case
Alfvén waves

Inertial waves

Internal gravity waves

Mixed waves:
Magneto-gravito-inertial (rotation and magnetic fields cannot be treated as perturbations)

Rotation and magnetic fields are perturbations

Figure 5
Wave types in a rotating and magnetized stably stratified radiative zone, with associated frequencies; \( \Omega \) is the Lamb frequency for an acoustic wave with degree \( l \), and \( V_A \) and \( k \) are the Alfvén speed and wave vector, respectively.

of a poloidal field, IGWs are trapped along poloidal field lines. Clearly, this affects low-frequency waves more profoundly than the high-frequency standing modes used to diagnose stellar interiors. These low-frequency waves are the most relevant for the transport of angular momentum.

3.4.3. Transport. In the absence of meridional flows and magnetic fields, the transport of angular momentum by waves is given by a horizontal average of the azimuthal component of the momentum equation (Zahn et al. 1997):

\[
\frac{d}{dr} \left( r^2 \rho \right) = - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho \sin \theta v_r v_\phi \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho v^2 r^4 \Omega_1 \right),
\]

where \( v_r \) and \( v_\phi \) denote horizontal averages with \( \Omega = \int_0^\pi \Omega \sin^2 \theta d\theta / \int_0^\pi \sin^3 \theta d\theta \) (Zahn 1992). This equation shows that the mean zonal flow is accelerated/decelerated by the divergence of the horizontally averaged Reynolds stresses \( F_\Omega (r) = \rho r \sin \theta v_r \Omega_1 \), often called the Eliassen–Palm flux (Eliassen & Palm 1960), and damped by viscous dissipation. The local characteristic timescale of the wave-induced transport is given by \( t_w = (\rho r^2 \Omega_1) / [1/r^2 \partial, (r^2 F_\Omega)] \). Fuller et al. (2014) found it to be shorter than \( 10^5 \) years for low-mass stars.

In a global sense, angular momentum transport by IGWs can couple convective and radiative regions (Talon et al. 2002, Rogers et al. 2013, Tayar & Pinsonneault 2013, Fuller et al. 2014, Rogers 2015, Pinçon et al. 2017). Hence, there is major interest in these waves now that the asteroseismic results in Figure 4 require efficient angular momentum coupling between such regions. The Doppler shift for IGWs is

\[
\omega(r) = \omega_{\text{gen}} - m[\Omega(r) - \Omega_{\text{gen}}],
\]

where \( \omega_{\text{gen}} \) and \( \Omega_{\text{gen}} \) are the frequency of excitation and the angular velocity at the excitation radius, and \( \omega(r) \) and \( \Omega(r) \) are the frequency and rotation rate in the local reference frame of the wave (Fuller et al. 2014). In the case of weak differential rotation \( \Delta \Omega \equiv \Omega(r) - \Omega_{\text{gen}} < \omega_{\text{gen}}/m \),
IGWs sustain shear until $\Delta \Omega \geq \omega_{\text{gen}}/m$. In the case of stronger differential rotation, critical layers develop and these reduce the differential rotation until $\Delta \Omega = \omega_{\text{gen}}/m$. From this, we immediately deduce that the largest differential rotation that can be tolerated in the presence of IGWs is $\Delta \Omega \sim \omega_{\text{gen}}/m$. For waves generated by turbulent convection in a star, the wave spectra of $\omega$ and $m$ are unknown, which makes it difficult to estimate this limiting differential rotation.

As an example, consider a main-sequence star in which the convective core is rotating faster than its radiative envelope, i.e., $\Delta \Omega < 0$. Assuming the convection generates prograde and retrograde waves equivalently, the prograde waves ($m > 0$) will be Doppler shifted to higher frequencies (if $\Delta \Omega < 0$), making them less susceptible to radiative damping as explained in Section 3.4.2. These can hence propagate further into the radiative region, where they deposit positive angular momentum when they dissipate. However, retrograde waves ($m < 0$) are shifted to lower frequencies (if $\Delta \Omega < 0$) and dissipate closer to the core, spinning down the region, and, more importantly, transporting less angular momentum because of the large density difference between the regions. The outer regions will hence spin up more than the inner regions will spin down. In this way, the convective core is coupled to the radiative envelope from the outside inward. In that case, the transport enforces weak differential rotation. In conclusion, though waves can couple convective and radiative regions, it is expected that these waves cause some positive or negative differential rotation within the radiative regions of stars. Therefore, as in the case of the advection by meridional flows and of the magnetic stresses, the transport of angular momentum by IGWs cannot be modeled as a diffusive process.

### 3.4.4. Application to stars

The first studies of the transport of angular momentum induced by stochastically excited IGWs were carried out for the Sun (Schatzman 1993, Kumar & Quataert 1997, Zahn et al. 1997, Talon & Charbonnel 2005). Talon & Charbonnel (2005) showed that IGWs can couple the convective envelope and radiative interior as well as bring about the uniform rotation of the radiative solar core. However, more detailed analyses of IGW transport in the solar radiative region (Rogers & Glatzmaier 2006, Denissenkov et al. 2008) indicated that these waves drive weak radial differential rotation, which would be detectable by helioseismology, yet this is not observed. Therefore, though IGWs can couple the convective and radiative regions in the Sun, another mechanism seems necessary to explain the uniform rotation profile of the bulk of the solar radiative interior up to 0.25 $R_\odot$. It is still unclear whether the core of the Sun rotates at the same rate as the bulk of the radiative interior, as both García et al. (2007) and Fossat et al. (2017) show an increased rotation rate in the core.

In intermediate- and high-mass stars, IGWs are likely to have more dynamical consequences than those in low-mass stars, because the waves propagate into a region of decreasing density, allowing the amplitudes of some waves to be amplified. Due to these increased amplitudes, some waves may break and interact with critical layers leading to enhanced angular momentum transport. Furthermore, the thermal diffusivity increases rapidly toward the surface of the star, which also leads to efficient damping and angular momentum transport. These effects have been demonstrated in 2D simulations of an equatorial slice for a 3-$M_\odot$ main-sequence star (Rogers et al. 2013, Rogers 2015). Depending on the strength of the convective flux used for the wave generation and of the level of rotation, several regimes for the resulting differential rotation can be obtained. In cases with high rotation and high forcing, the surface can spin faster than the core in the same direction. In cases with high rotation and low forcing or low rotation and low forcing, uniform rotation is obtained. In cases with low rotation and high forcing, retrograde differential rotation can be driven. Although incomplete in many ways, these numerical results agree with observations of differential rotation in the main-sequence stars shown in Figure 4. This was a major motivation...
Figure 6

(a) 3D visualization of simulations of IGWs from Edelmann et al. (2019), where the color scale shows temperature fluctuations from the background state. (b) Frequency spectra of the vertical velocity $v_r$ for IGW with $l = 2$ throughout a 3-$M_\odot$ ZAMS star. The vertical white lines are the frequencies of coherent $l = 2$ modes of the stellar model for the indicated radial order. Abbreviations: IGW, internal gravity wave; ZAMS, zero-age main sequence.

to generalize the 2D simulations by Rogers et al. (2013) into 3D simulations, with very similar results in terms of wave spectrum properties and angular momentum transport. A snapshot of the 3D temperature fluctuations is shown in Figure 6a, whereas the comparison between the quadrupole component of the generated IGW spectrum and the coherent quadrupole modes is shown in Figure 6b (Edelmann et al. 2019). The tangential velocities and temperature fluctuations resulting from these new large-scale long-term simulations are shown in Supplemental Figure 5. Observed frequency values due to coherent g modes or IGWs are dominantly determined by the near-core region for intermediate-mass and high-mass stars, because their amplitudes and hence probing power are dominant there. These fluctuations due to the IGWs are currently being used for the generation of synthetic light curves and line-profile variations to evaluate their relevance in explaining low-frequency power excess detected in space photometry (Tkachenko et al. 2014, Bowman et al. 2019) and signatures of macroturbulence observed in high-mass stars (e.g., Simón-Díaz & Herrero 2014), for which oscillations have been invoked as a physical explanation (Aerts et al. 2009).

In low-mass evolved stars, the efficiency of the transport of angular momentum driven by IGWs has been studied for the subgiant phase by Pinçon et al. (2017), who showed that IGWs can extract enough angular momentum to reduce the differential rotation induced by the stellar core contraction. However, IGWs are not efficient enough to explain the weak core rotation of red giants revealed in Figure 4 because of the strong thermal dissipation induced by the high stable stratification in the core (Cantiello et al. 2014, Fuller et al. 2014). As an alternative Belkacem et al. (2015a,b) showed that the angular momentum driven by mixed modes can be a good alternative to explain the observed low rotation rates. See the sidebar titled Summary Points From Theory and Simulations.
SUMMARY POINTS FROM THEORY AND SIMULATIONS

1. The zoo of hydrodynamical instabilities occurring in rotating stars is worth revisiting with the aim to calibrate them from asteroseismology; this will lead to a better understanding of their nonlinear properties and behavior as stars evolve.

2. The various theoretical prescriptions of convective penetration and core overshooting can be evaluated from their signature on g modes of single and binary intermediate-mass and high-mass stars; this will lead to an asteroseismically calibrated temperature gradient and mixing profile in the overshoot zone.

3. The theory of angular momentum transport by magnetism based on proper treatment of the Lorentz force should be revisited and included in stellar evolution codes. This can lead to predictions of observables to be tested against asteroseismology of magnetic pulsators of various kind (e.g., Kurtz 1990, Buysschaert et al. 2018).

4. Simulations show that angular momentum transport by IGWs offer great potential to explain the asteroseismic results for $\Omega(r)$ observed in a variety of stars (Figure 4). Strong efforts should be pursued to obtain good predictions for their excitation spectra, their propagation, and their damping in rotating (magnetized) stars.

4. FUTURE OUTLOOK TO IMPROVE STELLAR MODELING

Space asteroseismology opened a new window to the Universe by providing direct measurements of stellar interiors. As such, it revived the theory of stellar structure and evolution, pointing out major challenges for theories that remained uncalibrated so far but have been used in astrophysics for decades. A prominent shortcoming is a proper treatment of angular momentum transport in stellar models.

Although the results in Figure 4 are certainly suggestive, a drawback preventing us from deducing a causal connection between the core-hydrogen burning stars and their descendants is the lack of an accurate age, aside from the far too small sample sizes. The asteroseismic gravity, while of good precision, is too rough a proxy for the evolutionary status. Therefore, we need to populate Figure 4 with thousands of single and binary stars, covering the pre–main sequence up to the supergiant phase, with suitable and identified nonradial oscillation modes. Coming back to Figure 1, we have highlighted the major assets and limitations of 1D stellar modeling, keeping in mind the observational diagnostics and their level of precision in Table 1 as well as the current samples in Figure 4.

Improvements in the theory, following a data-driven approach, are within reach from existing Kepler/K2 and Gaia DR2 data. The upcoming Gaia DR3 and TESS (Transiting Exoplanet Survey Satellite) data will reveal thousands of suitable pulsators for asteroseismology in the Milky Way and in the Large Magellanic Cloud by the end of 2020. However, probing of $\Omega(r)$, $D_{\text{mix}}(r)$, and other quantities in the deep stellar interior requires gravito-inertial or mixed modes, and their frequencies can only be deduced from data with a duration of at least one year (Aerts et al. 2018). This can be achieved for stars in the two continuous viewing zones of TESS. On a somewhat longer term, the space data to be assembled with the European Space Agency’s (ESA) PLATO (PLAnetary Transits and Oscillations of stars) mission (to be launched in 2026; Rauer et al. 2014) will deliver the necessary frequency precision for samples of thousands of single and binary stars covering wide ranges in mass, metalicity, and rotation, including pre–main sequence stars and stellar clusters. This big data revolution in stellar astrophysics will not only allow us to populate Figure 4 but also replace the log $g$ by a high-precision seismic age estimate and construct similar diagrams for other quantities of critical importance for stellar interiors, such as the mixing profiles,
inside stars. This revolution will also allow us to finally identify and better model the dominant sources of angular momentum transport.

**FUTURE ISSUES**

1. The way forward in stellar modeling is to apply an integrated data-driven approach connecting and developing synergies between observations, new theoretical developments, and new 3D (M)HD simulations according to the spinner in Figure 1. The Kepler/K2 and Gaia data sets largely remain to be exploited with this aim.

2. A theoretical formulation of the optimal turbulent source term for wave generation in a star should be sought.

3. 3D (M)HD simulations covering various nuclear-burning phases should be performed; in order for them to be meaningful in terms of diagnostic predictions, these simulations must be global and large-scale in setup, covering the whole star and having long-duration time bases of several months, to cover the observed low-frequency regime observed for stars.

4. An aspect of Figure 1 that remained underexplored is tides in close binaries, the waves they trigger, and their impact on stellar interiors. The integration of tidal theory and binary asteroseismic modeling of observed stars should become a priority.

5. The development of 2D axisymmetric models that consider the deformation from spherical symmetry due to the centrifugal force (e.g., ESTER code by Rieutord et al. 2016) is of major importance to model stars rotating close to $\Omega_{\text{crit}}$. Upgrading the current 2D models by including the chemical evolution of the star, envelope convection, and angular momentum transport is needed to exploit asteroseismic data of the fastest rotators.

6. Current samples of core-hydrogen burning and hydrogen-shell burning stars with an asteroseismic measurement of $\Omega(r)$ are not representative in terms of mass, rotation, and binarity; those samples need to be extended appreciably to become unbiased.

7. Combined microarcsecond astrometry, high-resolution high signal-to-noise spectroscopy, and asteroseismic measurements of $\Omega(r)$ and $D_{\text{int}}(r)$ throughout the star, for single stars, close binaries, and clusters constitutes an optimal route to achieve a better theory of stellar evolution.

8. Asteroseismic measurements of $\Omega(r)$ are hardly available for high-mass stars and not at all for blue supergiants. The NASA TESS (launched April 2018) and ESA PLATO (to be launched in 2026) missions offer the opportunity to assemble the required space photometry and remedy this lack of stars for both Milky Way and Large Magellanic Cloud populations.

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LITERATURE CITED


Eddington AS. 1925. *Observatory* 48:73–75


Ferraro V. 1937. MNRAS 97:458–73


Leccanet D, Quataert E. 2013. *MNRAS* 430:2363–76
Ratnasingam RP, Edelmann PVF, Rogers TM. 2019. MNRAS 482:5500–12
Reed MD. 2016. IAU Focus Meet. 29:589–95
Salari S, Cassisi S. 2017. R. Soc. Open Sci. 4:170192
Shiode JH, Quataert E, Cantiello M, Bildsten L. 2013. MNRAS 430:1736–45